FOURIER METHODS

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First Edition

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PREFACE

This book is an introduction to Fourier series and Laplace transforms. Applications to physical problems involving ordinary and partial differential equations are included. The reader is assumed to have a working knowledge of elementary calculus, but where topics of advanced calculus are needed, they are developed from the beginning. Thus the discussion may help students and technologists to understand works in their field written in terms of harmonic analysis, complex exponentials, Fourier integrals, Fourier transforms, and Laplace transforms. The book is also suitable for use as a class text. Experience indicates that most of the topics can be covered in a one-semester course and that the material appeals particularly to applied mathematicians, engineers, and physicists.

The method of Fourier is interpreted here in a broad sense as referring to any analysis or synthesis of functions by a linear process applied to sines, cosines, or to complex exponentials. The initial chapter deals with complex quantities. It shows how to compute the elementary functions for complex values of the argument and how to read charts for finding such values approximately. It also explains what "complex impedance for a given frequency" means, and how to find it for a simple electrical or mechanical circuit.

The second chapter discusses averages and root mean square values as a preliminary to Fourier series. The series are found for simple functions by integration and for empirical functions numerically from a schedule for harmonic analysis. Fourier series are treated for the general interval, instead of for a period scaled down to 2π, which saves writing but often confuses the reader. After Fourier's theorem for periodic functions, full- and half-range series for functions on an interval are described. The

complex form of Fourier series leads to the Fourier integral theorem, Fourier transforms, and their relation to Laplace transforms.

As a preliminary to the solution of specific problems, Chap. 3 treats partial differential equations in a general way. Some simple general solutions as well as some particular solutions of a useful type are found. There is an explanation of the physical significance of the equations governing vibrations, the flow of heat, and the transmission of electricity, as well as Maxwell's equations and their relation to electrostatic fields and electromagnetic waves.

The following chapter solves boundary value problems for steady and variable heat flow, transmission lines, vibrating strings, and hollow wave guides. In several instances these applications are carried through to the evaluation of numerical results. For each type of application, a specific problem with numerical data is used to introduce the topic and to motivate the discussion, but eventually a general literal solution is derived.

In the final chapter the operational calculus is developed from the Laplace transform point of view and is applied to finding the transients in linear electrical and mechanical systems. The problems involve ordinary and partial differential equations, and in particular the method is used to find transient currents in electric networks and for the lossless transmission line. For easy reference, a table of Laplace transforms is reproduced on the inside of the back cover of the book.

Although there are many comprehensive treatments of some narrower aspect of Fourier methods, it is not easy for a reader with limited time available, or for a teacher giving a brief course, to extract from these the basic information which he needs. By concentrating on essentials, this volume enables the reader to gain a knowledge of Fourier methods in a broad sense, adequate for most applications.

There are thirty-one sets of practice problems, one for each major topic of the book. Answers to all problems are given at the end of the book.

References to alternative and to more extensive discussions are given in the final section of each chapter. A list of these works, with a few additions, is found in the Bibliography.

The author wishes to thank his colleague, Prof. Norman Levinson for valuable help and suggestions on the treatment of the Laplace transform.

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CAMBRIDGE, MASS.
December, 1949

CONTENTS

CHAPTER 1

COMPLEX QUANTITIES. IMPEDANCE

Expressions of the form $E \sin (\omega t + \phi)$ occur frequently in discussions of mechanical vibrations or of alternating currents in electrical networks. The calculations are simplified if we introduce complex quantities and work with the exponential of $i(\omega t + \phi)$, instead of using real trigonometric functions. In this chapter we shall review the algebra of complex numbers, define the exponential and trigonometric functions of complex quantities, and show how such functions may be used in combination with the notion of impedance to find the steady-state condition in an electrical or mechanical circuit.

1. Complex Quantities

A complex number is an expression of the form $a + bi$, where a and b are real numbers and i is the imaginary unit:

$$i = \sqrt{-1} \qquad \text{and} \qquad i^2 = -1 \tag{1}$$

Most of the rules for manipulating complex numbers are the same as those for real numbers. One useful principle is that, if a, b, a', and b' are all real, then the equation

$$a + bi = a' + b'i \qquad \text{implies that} \qquad a = a' \text{ and } b = b'. \tag{2}$$

Consequently, in any equation simplified to this form, we may equate the real and imaginary parts separately.

The addition of two complex numbers,

$$(a + bi) + (c + di) = (a + c) + (b + d)i, \tag{3}$$

is carried out just as if i were real. Likewise for subtraction,

$$(a + bi) - (c + di) = (a - c) + (b - d)i. \tag{4}$$

For multiplication, after multiplying the separate terms, there is one with i^2, which is replaced by -1 in accord with (1).   Thus

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i. \qquad (5)$$

For division, we may proceed as follows:

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (-ad + bc)i}{c^2 + d^2}$$
$$= \frac{ac + bd}{c^2 + d^2} + \frac{-ad + bc}{c^2 + d^2}\, i. \qquad (6)$$

In place of two real numbers a and b, which remain fixed throughout our discussion, we may use two real variables x and y, each of which takes on any one of some set or succession of values.   This leads us to consider the complex variable

$$z = x + iy.$$

The integral power function $w = z^n$ is defined by repeated multiplication.   From this, for any complex constant A_n, we may form $A_n z^n$, and set up polynomial expressions by adding together a finite number of such terms.

2. Exponential and Trigonometric Functions

To define the exponential, sine, and cosine functions for complex values of the variable, we use the infinite power series

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots , \qquad (7)$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots , \qquad (8)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots . \qquad (9)$$

These series are similar in form to the MacLaurin's series which represent the functions e^x, $\sin x$, and $\cos x$ for all real values of x. This shows that, when $y = 0$, so that $z = x + iy = x$, the values obtained from the new definition will agree with those previously used for real values of the variable.

The series (7), (8) and (9) converge for all complex values of z. Convergent series of this type may be multiplied and added together in the same way that polynomials are combined. It follows that the functions defined by the series satisfy the relation

$$e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}, \tag{10}$$

as well as the addition theorem for the sine

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \tag{11}$$

and that for the cosine

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2, \tag{12}$$

and the identity

$$\cos^2 z + \sin^2 z = 1. \tag{13}$$

Similarly, it follows from the series that

$$e^{iz} = \cos z + i \sin z \tag{14}$$

and

$$e^{-iz} = \cos z - i \sin z. \tag{15}$$

We may solve these for $\sin z$ and $\cos z$ and thus obtain

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \tag{16}$$

and

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{17}$$

The tangent, cotangent, secant, and cosecant are defined in terms of the sine and cosine by the quotients

$$\tan z = \frac{\sin z}{\cos z}, \qquad \cot z = \frac{\cos z}{\sin z} \tag{18}$$

$$\sec z = \frac{1}{\cos z}, \qquad \csc z = \frac{1}{\sin z}. \tag{19}$$

These are similar in form to relations of elementary trigonometry, which correspond to the special case $y = 0$, $z = x + iy = x$.

We may combine the definitions just given with Eqs. (16) and (17) to obtain expressions for tan z, cot z, sec z, and csc z in terms of exponentials.   For example,

$$\tan z = \frac{1}{i}\, \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i\, \frac{e^{2iz} - 1}{e^{2iz} + 1}. \tag{20}$$

Such expressions enable us to reduce any combination of trigonometric functions to a form involving exponentials only, and might have been used as the basis of trigonometry.   Specifically, we may take Eq. (7) as the definition of e^z and Eq. (10) as its fundamental property.   Then Eqs. (16), (17), and (20) and those similar to Eq. (20) may be used to define the six trigonometric functions.   The series (8) and (9) then follow from (16) and (17) combined with (7) and (1).   From this point of view Eqs. (11), (12), and (13) and all other trigonometric identities become a consequence of Eqs. (16), (17), and (7).

3. Derivatives

If $f(z)$ is a function of the complex variable $z = x + iy$, and while z is kept fixed,

$$\lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = f'(z), \tag{21}$$

regardless of how $h = \Delta z = \Delta x + i\,\Delta y$ approaches zero through *complex* values, we define $f'(z)$ as the derivative of $f(z)$ with respect to z, df/dz.   This definition is similar in form to that used for functions of a real variable in elementary calculus, but requires more of our function because we now require the limit to be the one value $f'(z)$ while Δy, as well as Δx, assumes any set of real values approaching zero.   However, as we shall illustrate presently, many of the rules of differential calculus carry over to the complex case.

The usual proof for real values, based on the binomial theorem, shows that for any positive integer n, and complex constant a,

$$\frac{d(az^n)}{dz} = anz^{n-1}. \tag{22}$$

Moreover, convergent power series may be differentiated term-wise. Consequently, it follows from the series (7), (8), and (9) that

$$\frac{d(e^z)}{dz} = e^z, \tag{23}$$

$$\frac{d(\sin z)}{dz} = \cos z, \tag{24}$$

$$\frac{d(\cos z)}{dz} = -\sin z. \tag{25}$$

Since the rule for differentiating composite functions

$$\frac{dw}{dz} = \frac{dw}{du} \cdot \frac{du}{dz} \tag{26}$$

also remains valid in the complex case, we may deduce that

$$\frac{d(e^{iz})}{dz} = ie^{iz} \qquad \text{and} \qquad \frac{d(e^{-iz})}{dz} = -ie^{-iz}. \tag{27}$$

We might have used these to derive Eqs. (24) and (25) from Eqs. (16) and (17).

EXERCISE I

Given $z_1 = 2 + 3i$, $z_2 = -3 - 2i$, $z_3 = 3 - 2i$, $z_4 = 26i$, find:

1. $z_1 + z_4$.   2. $z_1 - z_2$.   3. $z_2 + z_3$.   4. $z_3 - z_2$.
5. $z_1 z_2$.   6. $z_2 z_3$.   7. $z_2 z_4$.   8. z_1/z_3.
9. z_4/z_1.   10. z_4/z_2.   11. $z_3{}^2$.   12. $z_4{}^2$.

Use the power series (7) to check the tabular values

13. $e^{0.02} = 1.0202$.   14. $e^{0.08} = 1.0833$.   15. $e^{-0.06} = 0.94176$.

16. Using the values found in Probs. 13, 14, and 15, verify Eq. (10) for $z_1 = 0.08$ and $z_2 = -0.06$.

17. If n is any positive integer, show that

$$\cos nz = \tfrac{1}{2}(\cos z + i \sin z)^n + \tfrac{1}{2}(\cos z - i \sin z)^n.$$

From Prob. 17, with $n = 2, 3, 4, 5$, deduce that

18. $\cos 2z = \cos^2 z - \sin^2 z.$

19. $\cos 3z = \cos^3 z - 3 \cos z \sin^2 z.$

20. $\cos 4z = \cos^4 z - 6 \cos^2 z \sin^2 z + \sin^4 z.$

21. $\cos 5z = \cos^5 z - 10 \cos^3 z \sin^2 z + 5 \cos z \sin^4 z.$

22. If n is any positive integer, show that

$$\sin nz = -\tfrac{1}{2}i(\cos z + i \sin z)^n + \tfrac{1}{2}i(\cos z - i \sin z)^n.$$

From Prob. 22, with $n = 2, 3, 4, 5$, deduce that

23. $\sin 2z = 2 \cos z \sin z.$

24. $\sin 3z = 3 \cos^2 z \sin z - \sin^3 z.$

25. $\sin 4z = 4 \cos^3 z \sin z - 4 \cos z \sin^3 z.$

26. $\sin 5z = 5 \cos^4 z \sin z - 10 \cos^2 z \sin^3 z + \sin^5 z.$

Evaluate each of the following integrals after transforming the integrand to the second form by means of Eqs. (16), (17), (14), and (15):

27. $\int^x 8 \cos^2 x \sin^2 x \, dx = \int^x (1 - \cos 4x) dx.$

28. $\int^x 8 \cos^4 x \, dx = \int^x (\cos 4x + 4 \cos 2x + 3) dx.$

29. $\int^x 8 \sin^4 x \, dx = \int^x (\cos 4x - 4 \cos 2x + 3) dx.$

30. Prove that the indefinite integral $\int^x e^{(a+bi)x} \, dx = \dfrac{e^{(a+bi)x}}{a + bi}$, by using Eqs. (23) and (26) to differentiate the right member.

31. Assuming a, b, and x real, evaluate the two indefinite integrals $\int^x e^{ax} \cos bx \, dx$ and $\int^x e^{ax} \sin bx \, dx$ by equating the real and imaginary parts of the equation of Prob. 30.

4. Computation of the Functions

If $z = x + iy$, we find from Eqs. (10) and (14) that

$$\begin{aligned} e^z = e^{x+iy} = e^x \cdot e^{iy} &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y. \end{aligned} \tag{28}$$

This enables us to compute the value of e^z from tables of values of the real functions e^x, $\cos y$, and $\sin y$ with y in radian measure.

If y is not between 0 and $\pi/2 = 1.5708$, we may add or subtract multiples for $\pi/2$ to bring it in the range of the tables. For example, if $z = 2 + 3i$, we need $e^2 = 7.389$, $\sin 3$, and $\cos 3$. Since $3 - \pi = -0.1416$, we have $3 = \pi - (0.1416)$ and

$$\begin{aligned}
\sin 3 &= \sin 0.1416 = 0.1411, \\
\cos 3 &= -\cos 0.1416 = -0.9900.
\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}
e^{2+3i} &= e^2 \cos 3 + ie^2 \sin 3 = 7.389(-0.9900 + 0.1411i) \\
&= -7.315 + 1.042i.
\end{aligned} \tag{30}$$

To compute $\sin z$ and $\cos z$, we have, from Eqs. (11) and (12),

$$\sin z = \sin (x + iy) = \sin x \cos iy + \cos x \sin iy, \tag{31}$$
$$\cos z = \cos (x + iy) = \cos x \cos iy - \sin x \sin iy. \tag{32}$$

But from Eqs. (15) and (17), we have

$$\sin iy = \frac{e^{-y} - e^y}{2i} = i\left(\frac{e^y - e^{-y}}{2}\right), \tag{33}$$

$$\cos iy = \frac{e^y + e^{-y}}{2}. \tag{34}$$

This suggests that we tabulate the real functions

$$\sinh y = \frac{e^y - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^y + e^{-y}}{2} \tag{35}$$

read "hyperbolic sine" and "hyperbolic cosine." We may then compute the functions $\sin z$ and $\cos z$ from

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{36}$$
$$\cos z = \cos x \cosh y - i \sin x \sinh y. \tag{37}$$

5. Hyperbolic Functions

Let us study the hyperbolic functions defined by Eq. (35). Their graphs are shown in Fig. 1. It follows from Eqs. (33), (34), and (35) that

$$\sin iy = i \sinh y \quad \text{and} \quad \cos iy = \cosh y. \tag{38}$$

These relations may be used to deduce formulas for the hyperbolic

functions from those for the trigonometric functions. For example, if we replace z_1 by iy_1, and z_1 by iy_2 in Eq. (11), we find

$$\sin i(y_1 + y_2) = \sin iy_1 \cos iy_2 + \cos iy_1 \sin iy_2. \qquad (39)$$

But from this by Eq. (38), we may derive

$$i \sinh (y_1 + y_2) = (i \sinh y_1) \cosh y_2 \\ + \cosh y_1 (i \sinh y_2), \qquad (40)$$

so that, if we divide by i,

$$\sinh (y_1 + y_2) = \sinh y_1 \cosh y_2 \\ + \cosh y_1 \sinh y_2. \qquad (41)$$

In this way we show that

$$\cosh (y_1 + y_2) = \cosh y_1 \cosh y_2 \\ + \sinh y_1 \sinh y_2, \qquad (42)$$

and that

$$\cosh^2 y - \sinh^2 y = 1. \qquad (43)$$

FIG. 1. cosh x and sinh x.

By analogy with the defining relations (18) and (19), we define the hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, and hyperbolic cosecant by the equations

$$\tanh y = \frac{\sinh y}{\cosh y}, \qquad \coth y = \frac{\cosh y}{\sinh y}, \qquad (44)$$

$$\operatorname{sech} y = \frac{1}{\cosh y}, \qquad \operatorname{csch} y = \frac{1}{\sinh y}, \qquad (45)$$

The graph of tanh y is shown in Fig. 2. Each of the four functions just defined may be expressed in terms of exponential functions by using Eq. (35). For example,

$$\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}. \qquad (46)$$
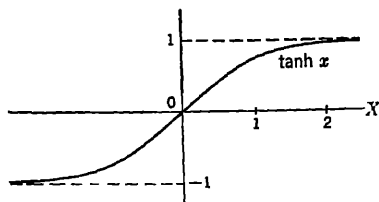
FIG. 2. tanh x.

For the derivatives of sinh y and cosh y, we may deduce from Eqs. (38), (24), (25), and (26) that

$$\frac{d(\sinh y)}{dy} = \cosh y \qquad \text{and} \qquad \frac{d(\cosh y)}{dy} = \sinh y. \quad (47)$$

These also follow directly from Eqs. (35) and (23).

The power series for $\sinh y$ and $\cosh y$ may be obtained by combining Eqs. (38) with (8) and (9), or more directly from Eqs. (35) and (7). The results are

$$\sinh y = y + \frac{y^3}{3!} + \frac{y^5}{5!} + \frac{y^7}{7!} + \cdots, \quad (48)$$

$$\cosh y = 1 + \frac{y^2}{2!} + \frac{y^4}{4!} + \frac{y^6}{6!} + \cdots, \quad (49)$$

All the above formulas involving hyperbolic functions of y hold for y complex, if we use these two series as the definitions of $\sinh y$ and $\cosh y$. For complex values, the hyperbolic functions may be computed from

$$\sinh (x + iy) = \sinh x \cos y + i \cosh x \sin y, \quad (50)$$
$$\cosh (x + iy) = \cosh x \cos y + i \sinh x \sin y. \quad (51)$$

If y is a large positive number, e^y is large and e^{-y} is small. By neglecting the e^{-y} terms in Eq. (35), we find that

$$\sinh y = \cosh y = \frac{e^y}{2}, \qquad \text{with a fractional error } e^{-2y}. \quad (52)$$

This shows that, with a small error for *large positive y*,

$$\log_{10} \sinh y = \log_{10} \cosh y = 0.434294y - 0.30103, \textit{ nearly.} \quad (53)$$

Since e^{-7} is a little less than 0.001, Eqs. (52) and (53) will give $\sinh y$ and $\cosh y$ correct to three figures if y exceeds 3.5, and correct to six figures if y exceeds 7. When y is beyond the range of our tables for $\sinh y$ and $\cosh y$, it is convenient to use Eq. (52) with a log log slide rule, or with a table for e^y when it is safe to interpolate in the exponential table. Otherwise Eq. (53) is useful. For example, if y in Eqs. (36) and (37) or x in Eqs. (50) and (51) were 12, we would need $\sinh 12$ and $\cosh 12$. They could be found from

$$\begin{aligned} \log_{10} \sinh 12 &= 12(0.434294) - 0.30103 = 4.91050, \\ \sinh 12 &= \cosh 12 = 81380. \end{aligned} \quad (54)$$

To find $\sinh y$ and $\cosh y$ when y is negative, we use

$$\sinh(-y) = -\sinh y, \qquad \cosh(-y) = \cosh y, \qquad (55)$$

which follow from Eqs. (48) and (49), or from Eq. (35).

EXERCISE II

Evaluate each of the following expressions in the form $a + bi$, with a and b each real numbers:

1. $e^{\pi i/3}$.　　　**2.** $e^{3\pi i}$.　　　**3.** $e^{-\pi i}$.　　　**4.** $e^{6\pi i}$.

5. $\cos 2i$.　　　**6.** $\sin 3i$.　　　**7.** $\cosh i$.　　　**8.** $\sinh 2i$.

9. e^{3-i}.　　　**10.** e^{-2+i}.　　　**11.** e^{4-2i}.　　　**12.** e^{7+10i}.

13. $\sin(1-i)$.　　　　　　**14.** $\cos(-1+2i)$.

15. $\sinh(3-2i)$.　　　　　**16.** $\cosh(-9+8i)$.

For hyperbolic functions, prove the following identities:

17. $1 - \tanh^2 x = \operatorname{sech}^2 x$.　　　**18.** $\coth^2 x - 1 = \operatorname{csch}^2 x$.

19. $\sinh 2x = 2 \sinh x \cosh x$.　　**20.** $\cosh 2x = \cosh^2 x + \sinh^2 x$.

21. $\sinh ix = i \sin x$.　　　　　　**22.** $\cosh ix = \cos x$.

23. $\cosh 3x = \cosh^3 x + 3 \cosh x \sinh^2 x = 4 \cosh^3 x - 3 \cosh x$.

24. $\sinh 3x = 3 \cosh^2 x \sinh x + \sinh^3 x = 4 \sinh^3 x + 3 \sinh x$.

Prove the following rules for differentiation:

25. $\dfrac{d(\tanh x)}{dx} = \operatorname{sech}^2 x$.

26. $\dfrac{d(\coth x)}{dx} = -\operatorname{csch}^2 x$.

27. $\dfrac{d(\operatorname{sech} x)}{dx} = -\tanh x \operatorname{sech} x$.

28. $\dfrac{d(\operatorname{csch} x)}{dx} = -\coth x \operatorname{csch} x$.

29. Check the tabular values $\sinh 0.3 = 0.3045$ and

$$\cosh 0.3 = 1.0453$$

a) by using the power series (48) and (49) and (*b*) by using Eq. 35) and a table of exponential functions.

30. In Fig. 3, with AB an arc of the unit circle $x^2 + y^2 = 1$ and the shaded area equal to $t/2$, show that $OC = \cos t$,

$$CB = \sin t,$$

and $AD = \tan t$. Similarly, in Fig. 4, with AB an arc of the unit rectangular hyperbola $x^2 - y^2 = 1$ and the shaded area equal to $t/2$, show that $OC = \cosh t$, $CB = \sinh t$, and $AD = \tanh t$. This analogy to the circular functions is the reason for the name
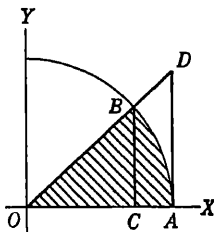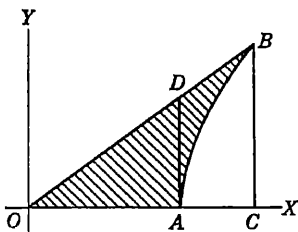
FIG. 3.   FIG. 4.

hyperbolic functions. HINT: Use polar coordinates r, θ, and call the shaded sector S. For the circle, $r = 1$, $S = \frac{1}{2}\theta$, so that $\theta = t$. For the hyperbola, right branch, if $y = \sinh u$, $x^2 = 1 + y^2$ makes $x = \cosh u$. Then $\theta = \tan^{-1} \frac{y}{x}$,

$$d\theta = \frac{x\, dy - y\, dx}{x^2 + y^2} = \frac{du}{r^2}.$$

Hence $dS = \frac{1}{2}r^2\, d\theta = \frac{1}{2}du$, $S = \frac{1}{2}u$, so that $u = t$.

31. If ϕ is in the interval $-\pi/2 < \phi < \pi/2$ and $\sinh x = \tan \phi$, show that $\cosh x = \sec \phi$, $\tanh x = \sin \phi$, $\coth x = \csc \phi$, $\csc h = \cot \phi$, $\operatorname{sech} x = \cos \phi$. Considered as a function of x, $\phi = \operatorname{gd} x$, is called the *Gudermannian* of x. By use of this function, and its inverse $x = \operatorname{gd}^{-1} \phi$, work involving hyperbolic functions can be treated with trigonometric functions and conversely.

32. If $x = 2$, compute ϕ from the defining relation of Prob. 31. Also verify that the values of the functions of x and ϕ taken from the tables satisfy the first two derived equations.

If $\phi = $ gd x and $x = $ gd^{-1} ϕ as in Prob. 31, show that

33. $\phi = 2 \tan^{-1} e^x - \pi/2$.   **34.** $x = \ln \tan \left(\dfrac{\pi}{4} + \dfrac{\phi}{2} \right)$.

35. $\int_0^\phi \sec \phi \, d\phi = x$.   **36.** $\int_0^x \operatorname{sech} x \, dx = \phi$.

37. If angle AOB of Fig. 4 is called θ, from Prob. 30 we have $\tan \theta = \tanh t$. Deduce from this that $\tan 2\theta = \sinh 2t$, so that, with the notation of Prob. 31, $\theta = \frac{1}{2}$gd $2t$.

6. The Logarithmic Function

If the natural logarithm of z, $\ln z$, is the function inverse to the exponential function, the relation

$$\ln (x + iy) = u + iv \tag{56}$$

implies that

$$x + iy = e^{u+iv} = e^u(\cos v + i \sin v), \tag{57}$$

where the last form is similar to Eq. (28).   Consequently

$$x = e^u \cos v \quad \text{and} \quad y = e^u \sin v, \tag{58}$$

since x, y, u, and v are all real.   It follows that

$$e^u = \sqrt{x^2 + y^2} \quad \text{and} \quad v = \tan^{-1}\frac{y}{x}, \tag{59}$$
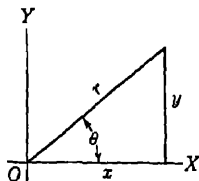
Fig. 5.

so that e^u and v may be used as the polar coordinates of a point in a plane with Cartesian or rectangular coordinates x and y. Let us denote a possible choice of polar coordinates with positive radius vector by r, θ. From Fig. 5,

$$r = \sqrt{x^2 + y^2}; \quad x = r \cos \theta; \quad y = r \sin \theta. \tag{60}$$

The number r is called the *absolute value* of $z = x + iy$, and we write

$$|z| = |x + iy| = r = \sqrt{x^2 + y^2}. \tag{61}$$

We find from Eqs. (58), (59), and (60) that

$$u = \ln r, \quad v = \theta, \tag{62}$$

leads to a value for the logarithm,

$$\ln(x + iy) = \ln r + i\theta. \tag{63}$$

Since θ is determined only to within an integral multiple of 2π, a complex number has an infinite number of possible logarithms. A particular value of θ may be determined from a knowledge of the sign of any two of the functions

$$\tan \theta = \frac{y}{x}, \qquad \cos \theta = \frac{x}{r}, \qquad \sin \theta = \frac{y}{r}, \tag{64}$$

and the numerical value of any one of the functions. Some computers use the tangent relation alone, but refrain from canceling any negative factors. Thus with this convention they would write for $x + iy =$

$$2 + 4i, \quad \tan \theta = 2, \qquad \theta = 1.1071,$$
$$-2 + 4i, \quad \tan \theta = \frac{2}{-1}, \quad \theta = \pi - 1.1071 = 2.0345$$
$$-2 - 4i, \quad \tan \theta = \frac{-2}{-1}, \quad \theta = \pi + 1.1071 = 4.2487 \text{ or } -2.0345$$
$$2 - 4i, \quad \tan \theta = \frac{-2}{1}, \quad \theta = 2\pi - 1.1071$$
$$= 5.1761 \text{ or } -1.1071 \tag{65}$$

The simplest method of checking a value of θ is to plot the point (x,y). The proper quadrant and a rough estimation of θ may then be read from the diagram. If θ_0 is any one value, all the possible values are given by

$$\theta = \theta_0 + 2k\pi, \qquad \text{where}$$
$$k = 0, 1, 2, \cdots \text{ or } -1, -2, \cdots \tag{66}$$

Suppose we superimpose on a square grid a series of concentric circles, each marked with a number equal to $\ln r$, where r is the radius, and a series of lines with constant θ, each marked with the value of θ. A few such lines and circles are drawn in Fig. 6, in which the square grid is omitted for clarity. Then by spotting the point x,y on the square grid, and after visual interpolation

reading the value of $\ln r$ and θ, we obtain the logarithm. For special problems, only a limited range will be of interest, and enlarged charts of this type may be made to give a fair degree of accuracy in the restricted range. Such a chart may be read in a reverse manner, starting with $\ln r$ and θ and reading x,y so as to obtain a value for the exponential function.
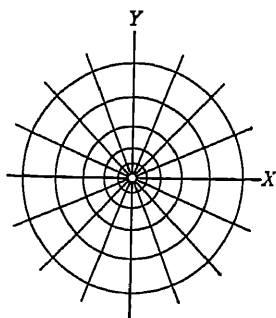
The rule for differentiating inverse functions,

$$\frac{dw}{dz} = \frac{1}{dz/dw} \qquad (67)$$

remains valid in the complex case. But

$$w = \ln z \qquad \text{implies that} \qquad z = e^w. \qquad (68)$$

Fig. 6.

Consequently, by Eq. (23),

$$\frac{dz}{dw} = e^w = z. \qquad (69)$$

It now follows from the last three numbered equations that

$$\frac{d(\ln z)}{dz} = \frac{1}{z}. \qquad (70)$$

We see from Eqs. (63) and (70) that the logarithmic function is defined and has a finite derivative for all finite values of z except $z = 0$. For $z = x + iy = 0$, $r = 0$ so that $\ln r = -\infty$ and θ is indeterminate. Hence we do not define $\ln z$ for $z = 0$.

The fundamental property of the logarithm

$$\ln (z_1 z_2) = \ln z_1 + \ln z_2 \qquad (71)$$

follows from Eqs. (68) and (10). It is true in the sense that, if any two of the logarithms are given, with particular choices of k in Eq. (66), some possible value of the third logarithm will make the equation true.

7. The Complex Plane

In the last section we were lead to associate the complex number $z = x + iy$ with the point P in a plane where $P = (x,y)$. We may think of the point P, or the vector OP, as representing the complex number. The operations of addition, subtraction, multiplication, and division on complex numbers then correspond to simple geometric operations on the vectors that represent them. We refer to P as the point z. Let

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2. \tag{72}$$

Then, by Eq. (3), for the sum

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2). \tag{73}$$

Thus OQ, the vector sum of OP_1 and OP_2 according to the parallelogram law, represents the algebraic sum of z_1 and z_2. Similarly
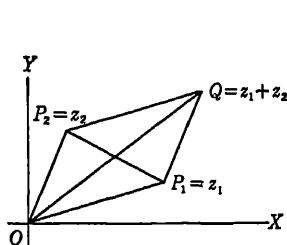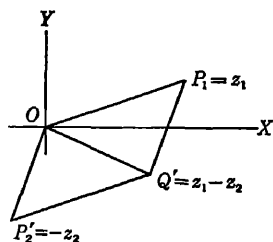
FIG. 7.　　　FIG. 8.

$z_1 - z_2$ is represented by the vector difference $OP_1 - OP_2$. That is, either OQ' in Fig. 8, obtained by adding OP_2 reversed or $OP_2' = -z_2$ to OP_1 vectorially, or P_2P_1 in Fig. 7.

For products and quotients, we introduce the polar coordinates shown in Fig. 5. Then, by Eqs. (60) and (14), we have

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) = r_1 e^{i\theta_1}, \tag{74}$$
$$z_2 = r_2(\cos \theta_2 + i \sin \theta_2) = r_2 e^{i\theta_2}. \tag{75}$$

It follows that for the product

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)} = r_1 r_2[\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)]. \tag{76}$$

Furthermore for the quotient

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} = \frac{r_1}{r_2} [\cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2)]. \quad (77)$$

These results also follow from Eqs. (63) and (71).

Equation (76) shows that, as in Fig. 9, if OP is the vector for the product and if OU is the vector to the point $1 = 1 + 0i$, then the triangle OP_1P is similar in sense and shape to the triangle OUP_2. Equation (77) shows that if OQ is the vector for the quotient z_1/z_2, then the triangle OP_1Q is similar in sense and shape to the triangle OP_2U. These facts may be used as the basis for a geometric method of multiplying and dividing complex numbers.
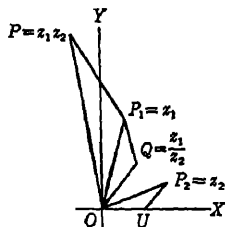
Fig. 9.

8. Powers and Roots

The relation

$$(e^w)^a = e^{aw} \quad (78)$$

is a consequence of Eq. (10) when a is an integer or rational number. For other values of a, real or complex, this equation may be used as the definition of the power in the left member. By Eq. (68), it follows that

$$\ln (z^a) = a \ln z. \quad (79)$$

Let us introduce the polar coordinates of z as in Eq. (74). Then

$$z = r(\cos \theta + i \sin \theta) = re^{i\theta}. \quad (80)$$

Also, from Eqs. (63) and (66),

$$\ln z = \ln r + i(\theta_0 + 2k\pi). \quad (81)$$

Hence

$$\ln (z^a) = a \ln z = a[\ln r + i(\theta_0 + 2k\pi)]. \quad (82)$$

If a is an irrational or complex number, each value of k will lead to a distinct value of z^a, and there will be an infinite number of values of the power.

If n is an integer, positive or negative, kn is an integer and

$$\ln (z^n) = n \ln r + in\theta = n \ln r + i(n\theta_0 + 2kn\pi). \qquad (83)$$

Thus only one value of the power is obtained, namely,

$$z^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta),$$
$$n = 0, 1, 2, \cdots \text{ or } -1, -2, \cdots. \qquad (84)$$

This is known as *De Moivre's theorem* and could have been obtained directly from Eq. (80).

If m is a positive integer, for the mth root or $(1/m)$th power we have

$$\ln(z^{1/m}) = \frac{1}{m} \ln r + i \frac{\theta}{m} = \frac{1}{m} \ln r + i \left(\frac{\theta_0 + 2k\pi}{m} \right). \qquad (85)$$

This makes

$$z^{1/m} = r^{1/m} \left(\cos \frac{\theta_0 + 2k\pi}{m} + i \sin \frac{\theta_0 + 2k\pi}{m} \right), \qquad (86)$$

m is a positive integer, $k = 0, 1, 2, \cdots, m - 1$.

This choice leads to m distinct values, and every other integral value leads to m distinct values, and every other integral value of k leads to a value of the root equal to one of these. Thus a complex number ($\neq 0$) has m distinct mth roots.

In calculating products, quotients, powers, and roots by Eqs. (76), (77), (84), and (86) it is convenient to have a short expression which can be written on one line to represent

$$e^{i\theta} = \cos \theta + i \sin \theta, \qquad (87)$$

such as cis θ, read "sis" θ, or $/\theta$, read "angle" θ. The notation $/\overline{\theta}$, read "lag angle" θ, is sometimes used in place of $/-\theta$. In any of these expressions, we may measure θ, or some part of θ, in degrees. For example, as one factor of a sixty cycle alternating current we might have

$$\underline{/377t + 20°} = e^{i(377t+20°)}$$
$$= \cos (377t + 20°) + i \sin (377t + 20°). \qquad (88)$$

Here the term $377t$ is in radians, which facilitates differentiation, while the phase 20° is kept in degrees, more convenient than radians when finding sines and cosines. If bothered by the departure from radian measure, the reader may think of the degree sign as an abbreviation for the factor $\pi/180 = 0.01745$. Using the modified notation just discussed, we may rewrite Eq. (86) as

$$z^{1/m} = r^{1/m} \operatorname{cis} \frac{\theta_0 + k360°}{m} = r^{1/m} \Big/ \frac{\theta_0 + k360°}{m}, \qquad (89)$$

m is a positive integer, $k = 0, 1, 2, \cdots, m - 1$.

Let us use this to compute the fourth roots of -16. We have

$$z = -16 = 16\underline{/180°}, \qquad r^{1/m} = 16^{\frac{1}{4}}$$

$$= 2, \qquad \frac{\theta_0}{m} = \frac{180°}{4} = 45° \quad (90)$$

Since $360°/m = 90°$, the four values of θ in Eq. (89) are

$$45°, \ 135°, \ 225°, \ 315°.$$

Thus the four values of the fourth root of -16 are

$$2\underline{/45°} = \sqrt{2} + i\sqrt{2}, \qquad 2\underline{/135°} = -\sqrt{2} + i\sqrt{2},$$
$$2\underline{/225°} = -\sqrt{2} \mp i\sqrt{2}, \qquad 2\underline{/315°} = \sqrt{2} - i\sqrt{2}. \quad (91)$$

EXERCISE III

Find the absolute value and one possible value of the angle for each of the following complex numbers:

1. $7 = 7 + 0i$.   **2.** $3i = 0 + 3i$.   **3.** $-5i = 0 - 5i$.
4. $-5 + 4i$.   **5.** $2 - 9i$.   **6.** $-4 - 7i$.

7. If a is any *real* number, prove that $|e^{ia}| = 1$.
8. For a and b real, prove that $|e^{a+bi}| = e^a$.

Calculate one value of the logarithm of each of the following complex numbers:

9. $-1 = -1 + 0i$.   **10.** $i = 0 + 1i$.   **11.** $4 - 4i$.
12. $-4 - 3i$.   **13.** $3 + 4i$.   **14.** $2 - 3i$.

Evaluate each of the following algebraically, and check by carrying out a graphical construction:

15. $(5 - 3i) + (-9 + 3i)$.   **16.** $(-7 + 4i) - (3 + 7i)$.

17. $(6 + 2i)(-1 - i)$.   **18.** $(3 - 3i)^2$.

19. $\dfrac{5 - 2i}{-2 - 5i}$.   **20.** $\dfrac{10 + 15i}{2 + 3i}$.

21. $(-2) + (2/\underline{60°})$.   **22.** $(3/\underline{135°}) + (3/\underline{45°})$.

23. $(6/\underline{115°})(2/\underline{-25°})$.   **24.** $(20/\underline{5°})(5/\underline{20°})$.

25. $\dfrac{6/\underline{40°}}{3/\underline{15°}}$.   **26.** $\dfrac{20/\underline{-40°}}{5/\underline{50°}}$.

Find all the values of each of the following indicated roots:

27. $\sqrt{-25}$.   **28.** $\sqrt{16i}$.   **29.** $\sqrt{9 + i}$.   **30.** $\sqrt{6 - 2i}$.

31. $\sqrt[3]{-64}$.   **32.** $\sqrt[4]{16}$.   **33.** $\sqrt[5]{32i}$.   **34.** $\sqrt[6]{1}$.

35. Show that $i^i = (0.2079)(535.5)^k$,
$$k = 0, 1, -1, 2, -2, \cdots.$$

36. Show that the vector drawn from the origin which represents the complex number $E/\underline{\omega t + \phi}$, with t the time and E, ω, ϕ constant, has constant length and rotates with uniform angular velocity.

37. For any complex number z, show that the vectors drawn from the origin representing the mth roots lie at the vertices of a regular polygon of m sides with center of symmety at the origin.
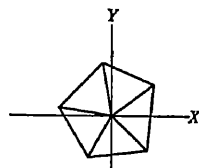
m values of $z^{1/m}$, for $m = 5$
FIG. 10.

9. Inverse Trigonometric Functions

If $\sin^{-1} z$ is the function inverse to the function $\sin z$, then

$$w = \sin^{-1} z \qquad \text{implies that} \qquad z = \sin w. \qquad (92)$$

Hence, by Eq. (16),

$$z = \frac{e^{iw} - e^{-iw}}{2i} \qquad \text{and} \qquad e^{iw} - 2iz - e^{-iw} = 0. \qquad (93)$$

We may write this in the form

$$(e^{iw})^2 - 2iz(e^{iw}) - 1 = 0, \tag{94}$$

a quadratic equation in e^{iw}, whose solution is

$$e^{iw} = iz + \sqrt{1 - z^2}. \tag{95}$$

For z complex there are two values of the square root, as seen from Eq. (86) with $m = 2$. When these are not real, there is no positive root. Hence the plus sign before the radical does not pick out a particular root, as it does for reals. This is why we write $+$ instead $\pm$. It follows from Eqs. (95) and (92) that

$$\sin^{-1} z = -i \ln (iz + \sqrt{1 - z^2}). \tag{96}$$

By a similar procedure we may deduce from Eq. (17) that

$$\cos^{-1} z = -i \ln (z + \sqrt{z^2 - 1}), \tag{97}$$

and from Eq. (20) that

$$\tan^{-1} z = \frac{i}{2} \ln \frac{1 - iz}{1 + iz}. \tag{98}$$

The rules of differentiation for these functions are similar in form to those which hold when the variables are real, namely,

$$\frac{d(\sin^{-1} z)}{dz} = \frac{1}{\sqrt{1 - z^2}}, \quad \text{where } \sqrt{1 - z^2} = \cos (\sin^{-1} z), \tag{99}$$

$$\frac{d(\cos^{-1} z)}{dz} = \frac{-1}{\sqrt{1 - z^2}}, \quad \text{where } \sqrt{1 - z^2} = \sin (\cos^{-1} z), \tag{100}$$

$$\frac{d(\tan^{-1} z)}{dz} = \frac{1}{1 + z^2}. \tag{101}$$

There may be derived either from Eqs. (96), (97), and (98) or by using Eq. (67) and the relations of the type of Eq. (92). The latter method of reasoning shows that in Eq. (99) we must take $\sqrt{1 - z^2} = \cos w$. This is also true of Eq. (96), since putting $z = \sin w$, and the radical equal to $\cos w$ makes the right member $w + 2k\pi$, while using $-\cos w$ for the radical makes the right member $\pi - w + 2k\pi$. Similarly the radical in Eq. (100) should

equal sin w, and that in Eq. (97) should equal i sin w.   If only one of the functions sin w, or cos w, is known, either value of the radical in Eq. (96), or in Eq. (97), gives a possible value of w.   If both sin w and cos w are known, w is determined to within a multiple of 2π and a specific value of the radical is given by the rules just stated which make the expressions in parentheses in both Eq. (96) and (97) reduce to cos $w + i$ sin w.

We shall next prove some further consequences of Eq. (92) by a discussion similar to that of the logarithmic function given in Sec. 6.   If $w = u + iv$, it follows from Eqs. (92) and (36) that

$$x + iy = z = \sin w = \sin u \cosh v + i \cos u \sinh v. \quad (102)$$

Consequently, since x, y, u, and v are all real,

$$x = \sin u \cosh v, \qquad y = \cos u \sinh v. \quad (103)$$

Suppose we select a number of evenly spaced values of v, all positive, and of u, all in the range from 0 to $\pi/2$, calculate the corresponding values of x and y, and plot the resulting points. We may then join the points with the same u by a curve, marked with this value of u, and similarly join the points with the same v, marked with this value of v.   Taking u in the range $\pi/2$ to π leads to the mirror image of these curves in the x axis, while the ranges $-\pi$ to $-\pi/2$ and $-\pi/2$ to 0 produce the reflections in the y axis of the curves for the corresponding positive values.   We could replace the values on the u curves by $\pi - u$, if at the same time we replaced v by $-v$.   Also, any value of u could be changed by $2k\pi$, k integral.   A few of the curves are shown in Fig. 11.   We observe that, similar to Eqs. (13) and (43),

$$\sin^2 u + \cos^2 u = 1 \qquad \text{and} \qquad \cosh^2 v - \sinh^2 v = 1. \quad (104)$$

The result of solving for sin u and cos u from Eq. (103) and substituting in Eq. (104) is

$$\frac{x^2}{\cosh^2 v} + \frac{y^2}{\sinh^2 v} = 1, \qquad \cosh^2 v - \sinh^2 v = 1. \quad (105)$$

This shows that the curves for constant v are ellipses with foci at

−1,0 and 1,0.   The result of solving for cosh v and sinh v from Eq. (103) and substituting in Eq. (104) is

$$\frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1, \qquad \sin^2 u + \cos^2 u = 1. \qquad (106)$$

This shows that the curves for constant u are branches of hyperbolas with foci at −1,0 and 1,0.
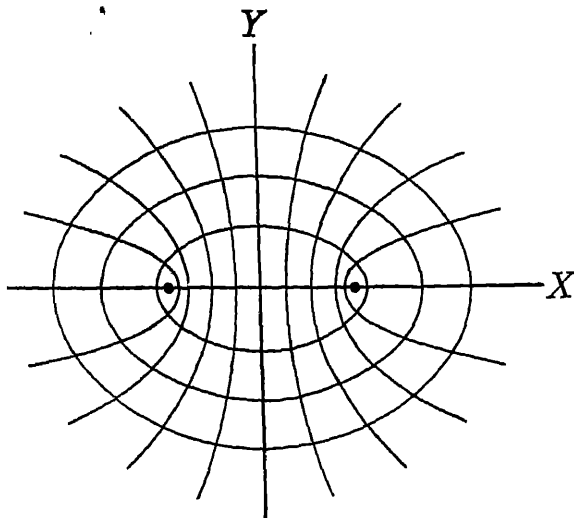
FIG. 11.

The curves just described, together with the x,y coordinate lines, constitute a chart for reading values of the inverse sine or sine.   For we may spot the point x,y and read u and v by visual interpolation between the values on the curves, thus obtaining $u + iv = \sin^{-1}(x + iy)$.   Or, starting with u,v, we may read x,y, thus obtaining $x + iy = \sin(u + iv)$.   A similar chart for the inverse cosine or cosine could be made by replacing each value of u on the sine chart by $\pi/2 - u$.

We may also derive a formula for computing $\sin^{-1} z$ which is sometimes preferable to Eq. (96).   The result of solving the

second part of Eq. (105) for $\sinh^2 v$, substituting in the first part and clearing of fractions, is

$$(\cosh^2 v)^2 - (1 + x^2 + y^2) \cosh^2 v + x^2 = 0. \qquad (107)$$

This is a quadratic equation in $\cosh^2 v$, whose solution is

$$\cosh^2 v = \tfrac{1}{2}[(1 + x^2 + y^2) + \sqrt{(1 + x^2 + y^2)^2 - 4x^2}], \qquad (108)$$

with the plus sign necessary to make $\cosh^2 v > 1$. This implies

$$\cosh v = \tfrac{1}{2}\sqrt{(1 + x)^2 + y^2} + \tfrac{1}{2}\sqrt{(1 - x)^2 + y^2}, \qquad (109)$$

as the square of the right member of Eq. (109) equals the right member of Eq. (108), and the sign is plus to make $\cosh v$ positive. And Eq. (109) and the first part of Eq. (103) imply that

$$\sin u = \tfrac{1}{2}\sqrt{(1 + x)^2 + y^2} - \tfrac{1}{2}\sqrt{(1 - x)^2 + y^2}, \qquad (110)$$

since the product of the right members of Eqs. (109) and (110) is x, which equals $\sin u \cosh v$, the product of the left members. We may use Eqs. (109) and (110) to calculate $\cosh v$ and $\sin u$, and hence $u + iv = \sin^{-1}(x + iy)$. If only the sine of $u + iv$ is known, we may take v plus or minus, and determine the quadrant of u so that the products in Eq. (102), or

$$\sin (u + iv) = \sin u \cosh v + i \cos u \sinh v \qquad (111)$$

have the proper signs. If the cosine of $u + iv$ is also known,

$$\cos (u + iv) = \cos u \cosh v - i \sin u \sinh v, \qquad (112)$$

by Eq. (37). Consequently, we know $-\sin u \sinh v$, which with $\sin u$ determines the sign of $\sinh v$, and hence the sign of v. The quadrant of u is again found from Eq. (111).

By an entirely parallel discussion based on Eq. (112) we may show that if $u + iy = \cos^{-1}(x + iy)$, then

$$\cosh v = \tfrac{1}{2}\sqrt{(1 + x)^2 + y^2} + \tfrac{1}{2}\sqrt{(1 - x)^2 + y^2}, \qquad (113)$$
$$\cos u = \tfrac{1}{2}\sqrt{(1 + x)^2 + y^2} - \tfrac{1}{2}\sqrt{(1 - x)^2 + y^2}. \qquad (114)$$

If only the cosine of $u + iv$ is known, we may take v plus or

minus and determine the quadrant of u so that the products in Eq. (112) have the proper sign. If the sine of $u + iv$ is also known, from Eq. (111) we know $\cos u \sinh v$, which with $\cos u$ determines the sign of $\sinh v$, and hence the sign of v. The quadrant of u is again found from Eq. (112).

To derive formulas for computing $u + iv = \tan^{-1}(x + iy)$, we first note that, by Eq. (98), this makes

$$2u + 2iv = i \ln \frac{1 + y - ix}{1 - y + ix}. \tag{115}$$

By Sec. 6, for the real part of the logarithm, we have

$$\ln|1 + y - ix| - \ln|1 - y + ix| = \frac{1}{2}\ln\frac{1 + x^2 + y^2 + 2y}{1 + x^2 + y^2 - 2y} \tag{116}$$

This is $2v$, and hence from Eq. (46) we find

$$\tanh 2v = \frac{2y}{1 + x^2 + y^2}. \tag{117}$$

By applying the method of Eq. (6), we have

$$\frac{1 + y - ix}{1 - y + ix} = \frac{1 - x^2 - y^2 - 2ix}{(1 - y)^2 + x^2}. \tag{118}$$

For the polar angle θ of this complex number, we have

$$\tan \theta = \frac{-2x}{1 - x^2 - y^2}, \quad \text{or} \quad \tan(-\theta) = \frac{2x}{1 - x^2 - y^2}. \tag{119}$$

Since the imaginary part of the logarithm in Eq. (115) is $i\theta$, $2u = i(i\theta) = -\theta$, and

$$\tan 2u = \frac{2x}{1 - x^2 - y^2}, \quad \sin 2u \text{ and } x \text{ same sign.} \tag{120}$$

The last remark is equivalent to applying the convention illustrated in Eq. (65) to the fraction in Eq. (120). It fixes the quadrant of $2u$.

In computing $u + iv = \tan^{-1}(x + iy)$ by Eqs. (117) and (120), the first determines $2v$ and hence v in both magnitude and

sign.   Equation (120) determines $2u$ to within a multiple of 2π, and hence u to within a multiple of π, if only the tangent of $u + iv$ is known.   If in addition we know either the sine or the cosine of $u + iv$, the quadrant of u may be fixed from any one of the terms in Eq. (111) or Eq. (112).

10. Inverse Hyperbolic Functions

A discussion like that of Sec. 9, based on Eqs. (35) and (46), shows that

$$\sinh^{-1} z = \ln (z + \sqrt{1 + z^2}), \tag{121}$$

$$\cosh^{-1} z = \ln (z + \sqrt{z^2 - 1}), \tag{122}$$

$$\tanh^{-1} z = \frac{1}{2} \ln \frac{1 + z}{1 - z}, \tag{123}$$

and that their derivatives are given by

$$\frac{d(\sinh^{-1} z)}{dz} = \frac{1}{\sqrt{1 + z^2}},$$
$$\text{where } \sqrt{1 + z^2} = \cosh (\sinh^{-1} z), \quad (124)$$

$$\frac{d(\cosh^{-1} z)}{dz} = \frac{1}{\sqrt{z^2 - 1}},$$
$$\text{where } \sqrt{z^2 - 1} = \sinh (\cosh^{-1} z), \quad (125)$$

$$\frac{d(\tanh^{-1} z)}{dz} = \frac{1}{1 - z^2}. \tag{126}$$

As indicated in the supplementary condition of Eq. (124), if $\sinh^{-1} z = w$, we must take $\sqrt{1 + z^2} = \cosh w$.   This is also true of Eq. (121), since putting $z = \sinh w$ and the radical equal to $\cosh w$ makes the right member $w + 2k\pi i$, while using $-\cosh w$ for the radical makes the right member $\pi i - w + 2k\pi i$.   Similarly, as indicated in Eq. (125), if $\cosh^{-1} z = w$, the radical in Eq. (125) and that in Eq. (122) should equal $\sinh w$.   If only one of the functions $\sinh w$, or $\cosh w$, is known, either value of the radical in Eq. (121), or Eq. (122), gives a possible value of w. If both $\sinh w$ and $\cosh w$ are known, w is determined to within a multiple of $2\pi i$ and a specific value of the radical is given by the

rules just stated which make the expressions in parentheses in both Eq. (121) and Eq. (122) reduce to $\cosh w + \sinh w = e^w$.

The problem of finding inverse hyperbolic functions of complex quantities may be reduced to the computation of inverse trigonometric functions by means of the relations

$$\sinh^{-1}(x + iy) = i\sin^{-1}(y - ix), \tag{127}$$
$$\cosh^{-1}(x + iy) = i\cos^{-1}(x + iy), \tag{128}$$
$$\tanh^{-1}(x + iy) = i\tan^{-1}(y - ix), \tag{129}$$

which follow from Eq. (38).

For z and w real, the positive root is always to be taken in Eqs. (121) and (124), since the hyperbolic cosine of a real quantity is always positive. For z real and greater than unity, the principal branch of $\cosh^{-1} z$ is taken as positive, so that for this positive branch the hyperbolic sine is positive and the positive root should be used in Eqs. (122) and (125).

EXERCISE IV

1. If $w = \sin^{-1} 2.6$, and $\cos w$ has a negative imaginary part, show that $\cos w = -2.4i$, and $\tan\left(\dfrac{\pi}{2} - w\right) = -\dfrac{2.4}{2.6}i$.

Compute the value of w in Prob. 1 by using

2. Eq. (96).   **3.** Eq. (97).
4. Eqs. (109) and (110).   **5.** Eqs. (113) and (114).
6. $\tanh i\left(\dfrac{\pi}{2} - w\right) = \dfrac{2.4}{2.6}.$ **7.** $z = \dfrac{2.6}{2.4}i$ in Eq. (98).

8. If $w = \cos^{-1} 5.05$, and $\tan w$ has positive imaginary part, show that $\sin w = 4.95i$, and $\tan w = 0.9802i$.

Compute the value of w in Prob. 8 by using

9. Eq. (96).   **10.** Eq. (97).
11. Eqs. (109) and (110).   **12.** Eqs. (113) and (114).
13. Eq. (98).   **14.** $\tanh(iw) = -0.9802.$

15. If $w = \sin^{-1}(1.2 + 0.4i)$, and $\cos w$ has a positive real part, show that $\cos w = 0.6 - 0.8i$, $\tan w = 0.4 + 1.2i$.

Compute the value of w in Prob. 15 by using

16. Eqs. (109 and (110).   **17.** Eqs. (117) and (120).

18. Prove that $\sinh^{-1} z = i \sin^{-1}(-iz)$, and use this to deduce Eq. (121) from Eq. (96).

19. Prove that $\cosh^{-1} z = i \cosh^{-1} z$, and use this to deduce Eq. (122) from Eq. (97).

20. Prove that $\tanh^{-1} z = -i \tan^{-1}(iz)$, and use this to deduce Eq. (123) from Eq. (98).

21. Check the tabular value $\cosh^{-1} 2 = 1.3170$ by means of Eq. (122).

22. Check the tabular value $\sinh^{-1} 0.5 = 0.4812$ by means of Eq. (121).

23. Check the tabular value $\tanh^{-1} 0.5 = 0.5493$ by means of Eq. (123).

For a and x real, prove the following integrations:

24. $\int^x \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a} \ln \dfrac{a + x}{a - x} = \dfrac{1}{a} \tanh^{-1} \dfrac{x}{a}$

25. $\int^x \dfrac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1} \dfrac{x}{a} = \ln (x + \sqrt{a^2 + x^2}) - \ln a$, where

   $\ln a$ may be absorbed in the integration constant.

26. $\int^x \dfrac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \dfrac{x}{a} = \ln (x + \sqrt{x^2 - a^2}) - \ln a$, where

   $\ln a$ may be absorbed in the integration constant.

27. Show that we may compute $u + iv = \sqrt{x + iy}$ from the equations $u^2 = \frac{1}{2}(x + \sqrt{x^2 + y^2})$, $v^2 = \frac{1}{2}(-x + \sqrt{x^2 + y^2})$, using such signs that uvy is positive when y is not zero.

28. Check the computation of $\cos w$ from $\sin w$ in Prob. 15, by using Prob. 27.

29. Fig. 12 shows the *catenary*, or

FIG. 12.

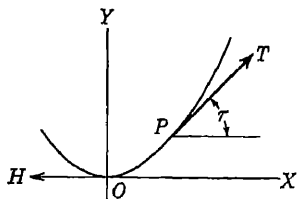curve of equilibrium of a chain or heavy flexible cable hanging

under its own weight, w lb. per foot of length s. If the tension is T lb. at $P = x,y$ and $\tan \tau = dy/dx = p$, and the tension is H lb. at $O = 0,0$ where $p = 0$, deduce from the equilibrium of arc OP that $T \cos \tau = H$ and $T \sin \tau = ws$. Hence, if $a = H/w$, $s/a = p$, and

$$a\, dp/dx = \sqrt{1 + p^2}.$$

By solving this for dx, and integrating as in Prob. 25, recalling that $x = 0$ when $p = 0$, deduce $x = a \sinh^{-1} p$ so that $\dfrac{dy}{dx} = \sinh \dfrac{x}{a}$ and hence $y = a \cosh \dfrac{x}{a} - a$.

30. Check the integration of $a\, dp/dx = \sqrt{1 + p^2}$ of Prob. 29 by putting $dp/dx = p\, dp/dy$, solving for y and integrating to obtain $y + a = a\sqrt{1 + p^2}$, so that $ap = \sqrt{(y + a)^2 - a^2}$. Put $p = dy/dx$, solve for dx and integrate as in Prob. 26 to obtain $x = a \cosh^{-1} (y + a)$ and hence $y = a \cosh \dfrac{x}{a} - a$.

31. For the catenary of Prob. 29, show that $s = a \sinh \dfrac{x}{a}$ and that $y = s \tanh \dfrac{x}{2a}$. For supports at the same level, s is the half length and x is the half span. Then $u = x/a$ is determined from $\dfrac{\sinh u}{u} = \dfrac{s}{x}$, using tables of the left member minus 1 for large sags, or for small sags replacing $\sinh u$ by $u + u^3/6$, the first two terms of Eq. (48). Then the sag $y = s \tanh \dfrac{u}{2}$, $a = x/u$, $H = wa$, and $T = H \sec \tau = H \cosh u$.

32. A wire weighing 0.02 lb./ft. is strung between two crossbars at the same level and 100 ft. apart. If the length of the wire is 101 ft., find the sag and the tension at the lowest point and at the ends (see Prob. 31).

33. The main cable of an aerial tramway weighs 8.8 lb./ft. For the first section, the lower end is 900 ft. below a point level with the upper end and at a horizontal distance of 3,000 ft. from it. If the lower end is horizontal when the cable supports no

load besides its own weight, find the tension at the two ends of
the first section and also the length and weight of this section.
HINT: With $u = x/a$, deduce from Prob. 29 that

$$\frac{y}{x} = \frac{\cosh u - 1}{u} = \frac{u}{2} + \frac{u^3}{24},$$

approximately, by Eq. (29). Omit u^3, and $u_1 = \dfrac{2y}{x} = 0.6$.

Then $u_2 = \dfrac{2y}{x} - \dfrac{u_1^3}{12} = u_1 \left(1 - \dfrac{u_1^2}{12}\right) = 0.582.$ An improved
value, 0.5833 is found by computing $\cosh u - 1 - 0.3u$ for u
near 0.582 and interpolating to make this zero. Then a, s, H, T
may be found from u by Prob. 31.

11. Simple Series Circuits

Suppose that electricity flows through an element containing
resistance, inductance, and capacity. We take one direction,
that from A to B in Fig. 13, as positive. If after t sec. the quan-
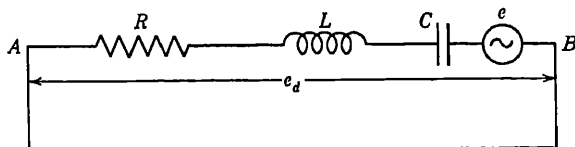
FIG. 13.

tity of electricity which has passed any point of the element in
the positive direction is q coulombs, the current intensity i
amperes for all points of the element at time t will be $i = dq/dt$.
The potential, or electromotive force (emf), at A minus that at B
will be the sum of three voltage drops diminished by the e volts
applied by the generating source. Thus the total drop

$$e_d = L \frac{di}{dt} + Ri + \frac{q}{C} - e. \tag{130}$$

The first drop is proportional to the rate of change of current
$di/dt = d^2q/dt^2$, and the positive constant of proportionality L

henrys is called the *inductance* of the element.   The second drop
is proportional to the current $i = dq/dt$, and the positive con-
stant of proportionality R ohms is the *resistance* of the element.
The third drop is proportional to the quantity of electricity
accumulated in the condensers since they were discharged at t_0
or $q = \int_{t_0}^{t} i\, dt$.   The positive constant of proportionality is taken
as $1/C$, and C farads is called the *capacity* of the element.

Let us next assume that points A and B are brought into con-
tact, or joined by an ideal conductor, so that they have the same
emf.   Then $e_d = 0$, and by combining the relation between q
and i with Eq. (130), we find

$$L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = e \qquad (131)$$

as the equation of the simple series circuit for q, and

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int_{t_0}^{t} i\, dt = e \qquad (132)$$

in terms of i.   Differentiation of this gives

$$L \frac{d^2i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = \frac{de}{dt}. \qquad (133)$$

Now consider a member of a mechanical system composed of a

Fig. 14.

mass attached to fixed members by a
spring and a dashpot.   Let s be the dis-
placement of the mass from equilibrium
taken positive when in the direction
from A to B in Fig. 14.   Then if F is a
force applied to an interior point of the
member, in the direction of AB, the
remaining external force F_e acting on
the member will be
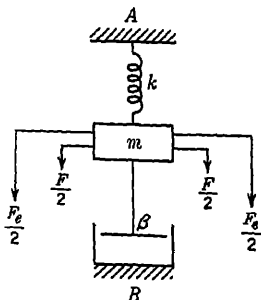
$$F_e = m \frac{d^2s}{dt^2} + \beta \frac{ds}{dt} + ks - F. \qquad (134)$$

We take the time t in seconds, s in feet, and the forces in pounds.

Then the constant of proportionality for the acceleration is m, the mass in slugs. That for the velocity, β lb.-sec./ft., measures the viscous resistance of the dashpot. And that for the displacement, k lb./ft., is the stiffness of the spring. The mass $m = w/g$, where w lb. is the weight and $g = 32.2$ ft./sec.² is the acceleration of gravity.

If F is the only force, $F_s = 0$, so that

$$m\frac{d^2s}{dt^2} + \beta\frac{ds}{dt} + ks = F \tag{135}$$

is the equation for the displacement. In terms of the velocity $v = ds/dt$ in feet per second this becomes

$$m\frac{dv}{dt} + \beta v + k\int_{t_0}^{t} v\,dt = F, \tag{136}$$

where t_0 is the time at which $s = 0$. Differentiation gives

$$m\frac{d^2v}{dt^2} + \beta\frac{dv}{dt} + kv = \frac{dF}{dt}. \tag{137}$$

A similar set of equations holds for an oscillating shaft,

$$I\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + K\theta = M. \tag{138}$$

Here I is the moment of inertia in slugs times square feet, or $mr^2 = wr^2/g$, where r is the radius of gyration. B measures the resistance of damping forces, K is the coefficient of torsion, and M ft.-lb. is the applied torque or moment. The variable θ is the angular displacement in radians. Other forms like Eqs. (136) and (137) may be obtained from this by introducing the angular velocity $\Omega = d\theta/dt$ radians per second. Thus

$$I\frac{d^2\Omega}{dt^2} + B\frac{d\Omega}{dt} + K\Omega = \frac{dM}{dt}. \tag{139}$$

12. Forced Vibrations

For the systems of Sec. 11, it is frequently important to know the response of an applied emf, force, or torque consisting of a

single term $E \sin (\omega t + \phi)$. Sometimes the general type of forcing term is a sum of terms like this, or an infinite series like the Fourier series of Chap. 2. In such cases we obtain the response for the sum by adding the responses for the separate terms. And there are times when the response to the sine term gives a sufficient indication of the behavior of the system for any forcing term of the same frequency.

We note that, by Eq. (87),

$$Ee^{i(\omega t+\phi)} = E\underline{/\omega t + \phi}$$
$$= E \cos (\omega t + \phi) + iE \sin (\omega t + \phi). \qquad (140)$$

In electrical-engineering literature, the mathematical symbols e and i are replaced by ϵ and j to reserve e for emf and i for current. With this notation

$$E\epsilon^{j(\omega t+\phi)} = E \cos (\omega t + \phi) + jE \sin (\omega t + \phi). \qquad (141)$$

This relation is also indicated by writing

$$E \cos (\omega t + \phi) = \mathrm{Re}\ E\underline{/\omega t + \phi} = \mathrm{Re}\ E\epsilon^{j(\omega t+\phi)}, \qquad (142)$$
$$E \sin (\omega t + \phi) = \mathrm{Im}\ E\underline{/\omega t + \phi} = \mathrm{Im}\ E\epsilon^{j(\omega t+\phi)}. \qquad (143)$$

The symbol Re, read "real part of," means the component along the real axis, while Im, read "imaginary part of," means the component along the $(i = j = \sqrt{-1})$ imaginary axis. Note that each produces a real value when applied to a complex number.

If we take the expression in Eq. (143) as the applied emf, e in Eqs. (132) and (133), they become

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_{t_0}^{t} i\,dt = E \sin (\omega t + \phi) = \mathrm{Im}\ E\epsilon^{j(\omega t+\phi)} \qquad (144)$$

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \mathrm{Im}\ jE\epsilon^{j(\omega t+\phi)}. \qquad (145)$$

To find a *particular integral* of this differential equation, we substitute in the left member

$$i = \mathrm{Im}\ YE\epsilon^{j(\omega t+\phi)}, \qquad (146)$$

where Y is a complex constant to be determined. We find

$$\text{Im } YEe^{j(\omega t+\phi)}\left[L(j\omega)^2 + R(j\omega) - \frac{1}{C}\right] = \text{Im } (j\omega)Ee^{j(\omega t+\phi)} \quad (147)$$

By Eq. (2), the imaginary parts will be equal if the complex numbers are equal, or if this equation holds when the symbol Im is omitted. That is, if

$$Y\left(Lj\omega + R + \frac{1}{Cj\omega}\right) = 1. \quad (148)$$

The expression in parentheses is called the *impedance* of the element for the frequency ω. It may be recalled by association with Eq. (144), and it may be obtained from the left member of that equation by replacing d/dt by $j\omega$ and replacing integration by $(j\omega)^{-1}$. We denote it by Z, and write

$$Z = Lj\omega + R + \frac{1}{Cj\omega} = R + j\left(L\omega - \frac{1}{\omega C}\right) = R + jX, \quad (149)$$

where $X = \text{Im } Z$ is the *reactance* of the element for the frequency ω. We also write, with the convention of Eq. (65),

$$Z = |Z| \underline{/\theta_z}, \text{ where } |Z| = \sqrt{R^2 + X^2}, \tan \theta_z = \frac{X}{R}. \quad (150)$$

Then we may deduce from Eqs. (148), (149), and (150) that

$$YZ = 1, \qquad Y = \frac{1}{Z} = \frac{1}{|Z|} \underline{/-\theta_z}. \quad (151)$$

With this value of Y, the particular integral of Eq. (146) is

$$i = \text{Im } YE \underline{/\omega t + \phi} = \text{Im } \frac{E \underline{/\omega t + \phi}}{Z} = \text{Im } \frac{E}{|Z|} \underline{/\omega t + \phi - \theta_z}$$
$$= \frac{E}{|Z|} \sin (\omega t + \phi - \theta_z). \quad (152)$$

In terms of the original constants, the particular integral is

$$\frac{E}{\sqrt{R^2 + \left(L\omega - \frac{1}{\omega C}\right)^2}} \sin \left[\omega t + \phi - \tan^{-1}\left(\frac{L\omega}{R} - \frac{1}{\omega RC}\right)\right], \quad (153)$$

where the inverse tangent has a positive cosine.

The expression just written remains a solution of Eq. (145) we add to it the *complementary function* which is the most genal solution of the equation with right member zero,

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = 0. \tag{154}$$

But ϵ^{rt}, or $c\epsilon^{rt}$, will be a solution of this equation if

$$L r^2 + R r + \frac{1}{C} = 0, \tag{155}$$

as we see by substituting $i = c\epsilon^{rt}$ in Eq. (154).   From this

$$r = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = -\frac{R}{2L} \pm j \sqrt{-\frac{R^2}{4L^2} + \frac{1}{LC}} \tag{156}$$

When $R^2 C$ exceeds $4L$, the radicand first written is positive, and its square root is less than $R/2L$.   Hence the roots are both real and negative, and we abbreviate them by $-f$, $-g$.   If $R^2 C = 4L$, the radicand is zero, and the two roots are each equal to $-a$, if $a = R/2L$.   In this case the left member of Eq. (155) is $L(r + a)^2$, and the left member of Eq. (154) is

$$L \left(\frac{d}{dt} + a \right) \left(\frac{d}{dt} + a \right) i = 0. \tag{157}$$

We verify that $ct\epsilon^{-at}$ is a second solution by noting that

$$\left(\frac{d}{dt} + a \right) t\epsilon^{-at} = \epsilon^{-at},$$

$$L \left(\frac{d}{dt} + a \right)^2 t\epsilon^{-at} = L \left(\frac{d}{dt} + a \right) \epsilon^{-at} = 0. \tag{158}$$

f $R^2 C$ is less than $4L$, the first radicand in Eq. (156) is negative, > that we use the second form.   With

$$a = \frac{R}{2L}, \qquad -\frac{R^2}{4L^2} + \frac{1}{LC} = b^2, \qquad r = -a + bj. \tag{159}$$

his leads to solutions of Eq. (154) which are multiples of

$$\epsilon^{rt} = \epsilon^{-at+jbt} = \epsilon^{-at} \cos bt + j\epsilon^{-at} \sin bt. \tag{160}$$

Since the right member of Eq. (154) is $0 = 0 + 0j$, the real and imaginary parts of the last expression in Eq. (160) must each separately be solutions, giving two real expressions.

The discussion just given shows that the *complementary function* for Eq. (145) may be written

$$c_1 \epsilon^{-ft} + c_2 \epsilon^{-gt} \qquad \text{when } R^2C > 4L, \qquad (161)$$
$$c_1 \epsilon^{-at} + c_2 t \epsilon^{-at} \qquad \text{when } R^2C = 4L, \qquad (162)$$
$$c_1 \epsilon^{-at} \cos bt + c_2 \epsilon^{-at} \sin bt \qquad \text{when } R^2C < 4L. \qquad (163)$$

And the complete solution of Eq. (145) is obtained by adding to the particular integral of Eq. (153) or Eq. (152) the appropriate form of the complementary function.

In an application to a specific situation, the values of the constants c_1 and c_2 could be found from two facts about the circuit, for example, the current at some one time i_1 at t_1 and the charge on the condenser at some one time q_2 at t_2. In this case, putting $t = t_1$ in the complete solution, and equating the result to i_1 would give one equation in c_1 and c_2. The second condition could be found by putting $t = t_2$ in the complete solution and its derivative, using the values so obtained for i and di/dt in Eq. (144), and replacing the integral in that equation by q_2. For example, if we knew i_0 the value of the current at $t = 0$, and the complementary function was given by the third form, Eq. (163), the first condition would be

$$\frac{E}{|Z|} \sin (\phi - \theta_z) + c_1 = i_0. \qquad (164)$$

And if at the same time $t = 0$ the charge on the condenser was zero, the second condition would be

$$\frac{L\omega E}{|Z|} \cos (\phi - \theta_z) - aLc_1 + bLc_2 + \frac{RE}{|Z|} \sin (\phi - \theta_z) + Rc_1 = E \sin \phi. \qquad (165)$$

For a real and positive, ϵ^{at} and ϵ^{at}/t become infinite when t becomes infinite. Hence ϵ^{-at} and $t\epsilon^{-at}$ approach zero and all the terms in the complementary function become small when t is very

large.   In fact, even after a fairly short time these terms become negligible in many applications.   Thus the important part of the solution in such cases is the particular integral, which represents the *steady-state*, or permanent, current.   The complementary function represents the *transient current*.   A convenient method of calculating transient currents will be given in Chap. 5.   To find the steady-state solution, we may proceed as follows.

Define *complex current* as the complex exponential term whose imaginary part is the actual simple harmonic steady-state current.   And define *complex* emf as the complex exponential term whose imaginary part is the actual applied simple harmonic emf, the left member of Eq. (141).   The impedance of the element, Z, for the frequency ω is defined in terms of L, R, C, and ω by Eq. (149).   Thus, for a given element, impedance is a complex variable which is a function of a real parameter, the frequency. Then, by Eq. (152), we have

$$\text{complex } i = \frac{\text{complex } e}{Z}, \quad \text{or} \quad iZ = e. \quad (166)$$

In the second form we write i and e, allowing their complex character to be inferred from the context, or the presence of the complex factor Z.   Equation (166) is similar in form to Ohm's law for direct currents.   It reduces the problem of finding the steady-state current in a single circuit to the problem of dividing two complex numbers and of taking the imaginary part of the quotient to obtain the real current.

The real part of the complex current, or quotient in Eq. (166), is the steady-state current due to an applied emf $E \cos (\omega t + \phi)$.

The similarity of Eq. (136) to Eq. (132) shows that if we define the *impedance Z* for the mechanical circuit by

$$Z = mj\omega + \beta + \frac{k}{j\omega} = \beta + j\left(m - \frac{k}{\omega}\right), \quad (167)$$

for frequency ω of the applied force

$$F = F_0 \sin (\omega t + \phi) = \text{Im } F_0 e^{j(\omega t + \phi)}. \quad (168)$$

The steady-state value of the velocity for the resulting forced vibration will be

$$v = \text{Im} \frac{\text{complex } F}{Z} = \frac{F_0}{|Z|} \sin (\omega t + \phi - \theta_z). \qquad (169)$$

Here the polar coordinates of Z are found from Eq. (167) to be

$$Z = \sqrt{\beta^2 + \left(m - \frac{k}{\omega}\right)^2}, \qquad \tan \theta_z = \frac{m - \dfrac{k}{\omega}}{\beta}. \qquad (170)$$

For the displacement s, we use a denominator built up from Eq. (135) with $j\omega$ in place of d/dt. Thus

$$s = \text{Im} \frac{\text{complex } F}{j\omega Z} = -\frac{F_0}{\omega|Z|} \cos (\omega t + \phi - \theta_z)$$

$$= \frac{F_0}{\omega|Z|} \sin \left(\omega t + \phi - \theta_z - \frac{\pi}{2}\right). \qquad (171)$$

13. Electric Networks

In an electric network consisting of several elements, the total drop in emf for each element may be expressed in terms of the current through that element, the constants for the element, and the applied emf in the element. For example, in Fig. 15 if element BC, numbered 1, has resistance R_1, inductance L_1, and capacity C_1 and the applied emf e_1 the potential at B minus that at C, e_{d1} is found as in Eq. (130) to be

$$e_{d1} = L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{q_1}{C_1} - e_1. \qquad (172)$$

The currents in the various elements of any network are related by Kirchhoff's first law, which states that

I. For all the elements which meet at any junction point, the algebraic sum of the currents, taken positive when toward the point, negative otherwise, is zero. This follows from the fact that, since current is conserved, as much goes out from any junction point as comes in.

The emfs in the various elements of any network are related by Kirchhoff's second law. This states that

· *II. For all the elements which make up any closed circuit, the algebraic sum of all the voltage drops for individual elements or differences of potential for successive junction points is zero.* For a plane network, it is convenient to go around the circuit clockwise.
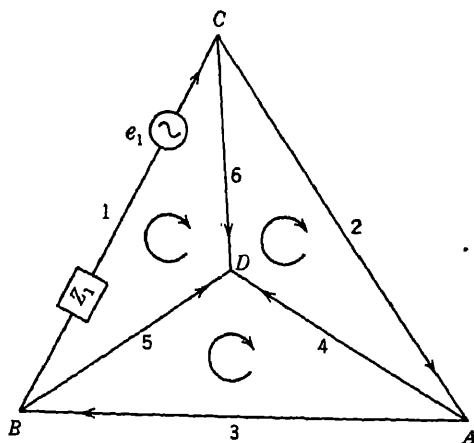
Fɪɢ. 15.

The second law expresses the fact that the total drop in emf must be zero when we return to the starting point.

To illustrate the application of these laws, consider the network of Fig. 15, with junction points A, B, C, D. We assign numbers to the elements, and positive directions, as indicated by the numbers and arrows in the figure. For the first element we have Eq. (172). And we may write five similar equations for the remaining elements by replacing the subscript 1 by 2, 3, 4, 5, or 6.

On applying law I to the points A, B, C, we find

$$i_2 - i_4 - i_3 = 0,$$
$$i_3 - i_5 - i_1 = 0,$$
$$i_1 - i_6 - i_2 = 0. \tag{173}$$

The three circuits marked by curved arrows in the figure are *fundamental* in the sense that any other circuit is a combination of one or more of them, with common elements traversed in opposite directions suppressed. On applying law II to them, we find

$$e_{d2} + e_{d4} - e_{d0} = 0,$$
$$e_{d3} + e_{d5} - e_{d4} = 0,$$
$$e_{d1} + e_{d0} - e_{d5} = 0. \tag{174}$$

It is possible, by introducing the impedances of the elements, to obtain the steady-state currents directly from Eqs. (173) and (174), as we shall show presently. However, to lead up to this short method, as well as to understand better the nature of these equations, we shall carry out certain further reductions. We first solve for i_4, i_5, and i_6 from Eq. (173), obtaining

$$i_4 = i_2 - i_3, \qquad i_5 = i_3 - i_1, \qquad i_6 = i_1 - i_2. \tag{175}$$

Now replace each of the six terms c_1 to e_6 in Eq. (174) by its value as given by Eq. (172) and the five similar equations. Next eliminate i_4, i_5, and i_6 by Eq. (175). And eliminate q_4, q_5, q_6 by equations similar in form to Eq. (175), obtained from it by integration from t_0 the time at which all condensers were discharged to t. Then Eq. (174) becomes

$$(L_2 + L_4 + L_6)\frac{di_2}{dt} + (R_2 + R_4 + R_6)i_2 + \left(\frac{1}{C_2} + \frac{1}{C_4} + \frac{1}{C_6}\right)q_2$$
$$-L_4\frac{di_3}{dt} - R_4 i_3 - \frac{1}{C_4}q_3 - L_6\frac{di_1}{dt} - R_6 i_1 - \frac{1}{C_6}q_1$$
$$= e_2 + e_4 - e_6, \tag{176}$$

and two other relations of similar form. Differentiating these equations, and replacing dq_1/dt by i_1, dq_2/dt by i_2, and dq_3/dt by i_3, we obtain three second-order differential equations in the currents i_1, i_2, i_3. After these are solved for these currents, the remaining three may be found from Eq. (175).

The discussion just given for Fig. 15 could be applied to any network. We would apply law I to all but one of the points of

junction, and we would then solve for as many of the currents as
we could in terms of the rest.   We would then apply law II to any
fundamental set of circuits and eliminate the currents for which
we solved the equations obtained from law I.   The result is
reducible to a system of differential equations, each at most of
the sec nd order, with as many equations as there are currents
to be determined.   For such a system the complementary func-
tion may be obtained by putting $i_1 = c_1 \epsilon^{rt}$, $i_2 = c_2 \epsilon^{rt}$, etc., in the
equations with right member, or combinations of the e terms,
replaced by zero.   The term ϵ^{rt} may be divided out, and elimi-
nation of the coefficients c_1, c_2, etc. (for example by setting the
determinant of the factors multiplying them, equal to zero) leads
to an equation $P(r) = 0$, where $P(r)$ is a polynomial.   Each root
of this equation leads to a set of values of ratios of the coefficients
c_1, c_2, etc., and hence to a term of the complementary function
for each i of the form $kc_1 e^{rt}$ for i_1.   The r and c_1, c_2, etc., are fixed,
and there is one arbitrary constant k in each i for each root r.
For an actual dissipative network, the roots r will either be real
and negative, or if complex have a negative real part.   Thus the
terms in the complementary function, like those of Eqs. (161),
(162), and (163), will each become small for t large.   Hence they
will correspond to transient currents.   The process just described
is chiefly of theoretical interest, because it is easier to find the
transient currents, with the constants evaluated for specific initial
conditions, by the method described in Chap. 5.

When only the steady-state solution is needed, we may proceed
as follows.   For a particular frequency present in one or more of
the applied emfs omit all terms not of this frequency.   With each
applied emf, associate a complex exponential of which it is the
imaginary part as

$$\text{complex } e_1 = E_1 \underline{/\omega t + \phi_1} = E_1 \epsilon^{j(\omega t + \phi_1)}. \tag{177}$$

And, when all the e's of this frequency are applied, if the current
response in element 1 is $I_1 \sin (\omega t + \psi_1)$, define

$$\text{complex } i_1 = I_1 \underline{/\omega t + \psi_1} = I_1 \epsilon^{j(\omega t + \psi_1)}. \tag{178}$$

Similar to Eq. (149), we define the impedance Z_1 by

$$Z_1 = R_1 + j\left(L_1\omega - \frac{1}{\omega C_1}\right). \tag{179}$$

Then by Eq. (172), the complex voltage drop is

$$e_{d1} = Z_1 i_1 - e_1, \qquad e_{d1}, e_1, i_1 \text{ complex.} \qquad \tag{180}$$

Definitions and relations for the other elements are similar to these. Now regard the terms of Eq. (174) as complex, and replace each term by its equivalent obtained from Eq. (180) and the equations similar to it. Then consider the terms in Eq. (175) as complex, and use these relations to eliminate the last three complex currents. This leads to

$$\begin{aligned}
(Z_2 + Z_4 + Z_6)i_2 - Z_4 i_3 - Z_6 i_1 &= e_2 + e_4 - e_6, \\
(Z_3 + Z_5 + Z_4)i_3 - Z_5 i_1 - Z_4 i_2 &= e_3 + e_5 - e_4, \\
(Z_1 + Z_6 + Z_5)i_1 - Z_6 i_2 - Z_5 i_3 &= e_1 + e_6 - e_5.
\end{aligned} \tag{181}$$

This system may be solved for any one of the currents. For example, in terms of determinants, we find for i_1

$$i_1 = \frac{\begin{vmatrix} e_2 + e_4 - e_6 & Z_2 + Z_4 + Z_6 & -Z_4 \\ e_3 + e_5 - e_4 & -Z_4 & Z_3 + Z_5 + Z_4 \\ e_1 + e_6 - e_5 & -Z_6 & -Z_5 \end{vmatrix}}{\begin{vmatrix} -Z_6 & Z_2 + Z_4 + Z_6 & -Z_4 \\ -Z_5 & -Z_4 & Z_3 + Z_5 + Z_4 \\ Z_1 + Z_6 + Z_5 & -Z_6 & -Z_5 \end{vmatrix}}. \tag{182}$$

For given numerical values of the constants, this could be reduced to a complex constant times $\epsilon^{j\omega t}$, and its imaginary part would give the current response of frequency ω in the first element.

For any network, the introduction of complex currents, complex emfs, and impedance for each element leads to relations like Eq. (180). And these may be combined with the equations obtained from Kirchhoff's laws I and II, regarded as holding for the complex terms, to obtain a system of simultaneous equations of the first degree which may be solved for the complex currents contained in them.

For simple mechanical circuits, the use of complex exponentials and impedance was described in Eqs. (167) to (171). The extension of these notions for more complicated mechanical systems is illustrated in Probs. 24 to 28 of Exercise V.

14. References

In the discussion of complex numbers and complex exponentials the formal side has been emphasized, the rules for manipulating power series being stated without proof. For a discussion with more emphasis on theoretical questions, the interested reader is referred to Chaps. V, IX, and XIII of the author's *Treatise on Advanced Calculus.*

Tables of hyperbolic functions and of trigonometric functions for arguments in radians are to be found in most mathematical handbooks. As more extensive than most such tables, H. B. Dwight's *Mathematical Tables* is recommended.

For additional information about the physical applications mentioned here, the reader may consult E. A. Guillemin's *Communication Networks*, S. Timoshenko's *Vibration Problems in Engineering*, or J. P. Den Hartog's *Mechanical Vibrations.*

EXERCISE V

1. Find the transient current when a condenser of capacity 5 microfarads $= 5 \times 10^{-6}$ farad charged with 0.006 coulomb is discharged through a circuit containing a resistance of 3 ohms and an inductance of 10 henrys.

2. Show that the steady-state solution for i given by Eq. (152) or Eq. (153) is always a solution of Eq. (145), but that it is a solution of Eq. (144) only when the term involving t_0, a multiple of $\cos (\omega t_0 + \phi - \theta_z)$, is zero.

A simple circuit has $R = 60$ ohms, $L = 8$ henrys, and

$$C = 3 \text{ microfarads} = 3 \times 10^{-6} \text{ farad.}$$

Find the steady-state current when the applied emf e is

3. 100 sin $(120\pi t)$.   **4.** 100 cos $(120\pi t)$.

5. $10 \sin (360\pi t + 40°)$.   **6.** $10 \sin (600\pi t - 20°)$.

7. $50 \sin (120\pi t) + 20 \sin (360\pi t + 40°)$.

8. $50 \sin (120\pi t) + 20 \sin (600\pi t - 20°)$.

A simple circuit has resistance R, inductance L, and capacity C. Find the steady-state current if e is

9. $\sin \omega t$.   **10.** $\cos \omega t$.

11. $A \cos \omega t + B \sin \omega t$.   **12.** $\cos (\omega t + \alpha)$.

13. Check Prob. 12, by using Prob. 11 with $A = \cos \alpha$ and $B = -\sin \alpha$.

14. An 8-lb. weight is constrained by a spring which stretches 1 ft. under a 5-lb. pull and a dashpot which offers 1 lb. resistance when the velocity is 1 ft./sec. Find the steady-state forced vibrations when the applied force $F = 20 \sin 4t$.

15. A heavy disk has a radius of gyration 1 ft. and weighs 100 lb. It vibrates about its axis due to an applied moment $M = 0.2 \sin 300t$. If it is constrained by a damper for which B is 10 ft.-lb.-sec. and a torsional constraint for which K is 20 ft.-lb., find the steady-state forced vibrations.

16. For a given frequency, let the elements of Fig. 16 have impedances Z_1, Z_2, Z_3 and applied complex emfs e_1, e_2, e_3. Use

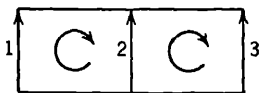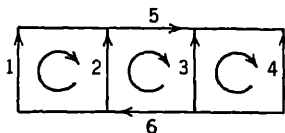Fig. 16.   Fig. 17.

Kirchhoff's laws to set up the equations which determine the complex currents.

17. In Prob. 16, let $Z_3 = 0$, $e_1 = 0$, $e_2 = 0$. Solve for i_3 and interpret your result to give the rule for finding the single impedance equivalent to two impedances in parallel,

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}, \quad \text{or} \quad Z = \frac{Z_1 Z_2}{Z_1 + Z_2}.$$

18. In Fig. 17, let the only applied emf be in element 6, and let this be the complex part of e_6. For the frequency of e_6, let the elements have impedances $Z_1, \cdots, Z_6$. Set up the equations which determine the complex currents.

19. Solve the equations of Prob. 18 for i_6, and check by finding the sum of the impedances for Z_1 and Z_2 in parallel, Z_3 and Z_4 in parallel, Z_5, and Z_6 (see Prob. 17).

20. In Fig. 16 let element 1 contain resistance R_1, inductance $(L_1 - M)$, and real applied emf e_1. Also let element 3 contain resistance R_3, inductance $(L_3 - M)$, and real applied emf e_3. If then element 2 contains inductance M, show that the equations for the real currents i_1 and i_2 are

$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + M \frac{di_3}{dt},$$

$$e_3 = R_3 i_3 + L_3 \frac{di_3}{dt} + M \frac{di_1}{dt}.$$

These are the equations for two circuits with mutual inductance M, and the result shows that the presence of mutual inductances in a network does not change the character of the equations. As in this case, a network with mutual inductances can always be replaced by one without them, but with some added elements and changed inductances, possibly to negative values.

21. If in Prob. 20, $e_1 = E_1 \sin t$ and $e_3 = E_3 \sin t$, find the steady-state solution for i_1.

22. In Fig. 17, let the only applied emf be in element 1, and let this be the complex part of e_1. Let the ends of element 6 be short-circuited so that $Z_6 = 0$. And for the frequency of e_1, let the other numbered elements have impedances $Z_1, \cdots, Z_5$. Show that the complex currents i_1, i_2, i_3 satisfy the following system of equations:

$$Z_1 i_1 + Z_4(i_1 - i_2) = e_1,$$
$$Z_4(i_2 - i_1) + Z_2 i_2 + Z_5(i_2 - i_3) = 0,$$
$$Z_5(i_3 - i_2) + Z_3 i_3 = 0.$$

23. Solve the system of Prob. 22 for i_1 and show that the self-

impedance of element 1, e_1/i_1, may be written

$$\frac{e_1}{i_1} = Z_1 + \cfrac{1}{\cfrac{1}{Z_4} + \cfrac{1}{Z_2 + \cfrac{1}{\cfrac{1}{Z_3} + \cfrac{1}{Z_5}}}}.$$

Check this by using Prob. 17 twice, first for Z_3, Z_5 in parallel, then this in series with Z_2 taken in parallel with Z_4.

24. The mechanical system of Fig. 18 consists of three vibrating masses, m_1, m_2, m_3, each attached by a spring and dashpot to a fixed base, as indicated by the elements 1, 2, 3. In addition there are spring and dashpot connections between the masses, as indicated by the elements 4 and 5. As in Eq. (167), we define impedances Z_1, Z_2, Z_3 by $Z_1 = m_1 j\omega + \beta_1 + \dfrac{k_1}{j\omega}$, and two similar equations. We also define $Z_4 = \beta_4 + \dfrac{k_4}{j\omega}$, $Z_5 = \beta_5 + \dfrac{k_5}{j\omega}$. The
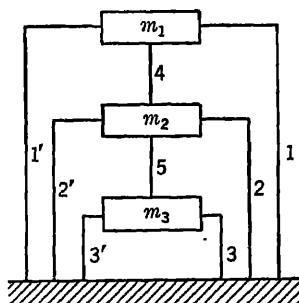
FIG. 18.

only applied force of frequency ω, acts on m_1, and is the complex part of F_1. Show that the complex velocities v_1, v_2, v_3 satisfy the following system of equations:

$$\begin{aligned}
Z_1 v_1 + Z_4(v_1 - v_2) &= F_1, \\
Z_4(v_2 - v_1) + Z_2 v_2 + Z_5(v_2 - v_3) &= 0, \\
Z_5(v_3 - v_2) + Z_3 v_3 &= 0.
\end{aligned}$$

25. Observing the similarity of the system of equations of Prob. 24 to that of Prob. 22, deduce from Prob. 23 that, for the complex velocity v_1 and displacement s_1 of m_1 in the mechanical system,

$$\frac{F_1}{v_1} = \frac{F_1}{j\omega s_1} = Z_1 + \cfrac{1}{\cfrac{1}{Z_4} + \cfrac{1}{Z_2 + \cfrac{1}{\cfrac{1}{Z_3} + \cfrac{1}{Z_5}}}}.$$

26. A shaft carrying three disks, Fig. 19, is undergoing tor-
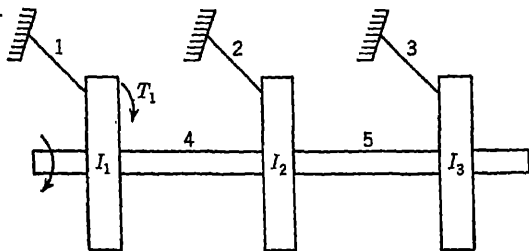
FIG. 19.

sional oscillations caused by a torque applied to the first disk. This torque is of frequency ω and is the complex part of T_1. With the notation of Eq. (138), for the first disk let I_1 be the moment of inertia, the angular displacement be the complex part of θ_1, and the angular velocity be the complex part of Ω_1. And let the element 1 denote a restraining tendency toward a neutral position with a spring constant K_1 and a damping restraint with constant B_1. This leads us to define the impedance

$$Z_1 = I_1 j\omega + B_1 + \frac{K_1}{j\omega}.$$

Similarly for disks 2 and 3, and the restraints indicated by elements 2 and 3, we define Z_2 and Z_3. The elements 4 and 5 denote restraints depending on the relative position and velocity of the disks, so that for them the impedances are $Z_4 = B_4 + \dfrac{K_4}{j\omega}$,

$_5 = B_5 + \dfrac{K_5}{j\omega}$. Show that Ω_1, Ω_2, Ω_3 satisfy the following

stem of equations:

$$Z_1\Omega_1 + Z_4(\Omega_1 - \Omega_2) = T_1,$$
$$Z_4(\Omega_2 - \Omega_1) + Z_2\Omega_2 + Z_5(\Omega_2 - \Omega_3) = 0,$$
$$Z_5(\Omega_3 - \Omega_2) + Z_3\Omega_3 = 0.$$

27. Observing the similarity of the system of equations of Prob. 26 to that of Prob. 22, deduce from Prob. 23 that, for the torsional system,

$$\frac{T_1}{\Omega_1} = \frac{T_1}{j\omega\theta_1} = Z_1 + \cfrac{1}{\cfrac{1}{Z_4} + \cfrac{1}{Z_2 + \cfrac{1}{\cfrac{1}{Z_3} + \cfrac{1}{Z_5}}}}.$$

28. Suppose that there are n meshes in Prob. 22, n masses in Prob. 24, or n disks in Prob. 26. Number the connections $n + 1$ from 1 to 2, $N + 2$ from 2 to 3, $\cdots$, $2n - 1$ from $n - 1$ to n. Show that the continued fraction for $\dfrac{e_1}{i_1} = \dfrac{F_1}{v_1} = \dfrac{F_1}{j\omega s_1} = \dfrac{T_1}{\Omega_1} = \dfrac{T_1}{j\omega\theta_1}$ is

$$Z_1 = \cfrac{1}{\cfrac{1}{Z_{n+1}} + \cfrac{1}{Z_2 + \cfrac{1}{\cfrac{1}{Z_{n+2}} + \cdots}}}$$
$$\cdots$$
$$\cdots + \cfrac{1}{Z_{n-1} + \cfrac{1}{\cfrac{1}{Z_{2n-1}} + \cfrac{1}{Z_n}}}.$$

CHAPTER 2

FOURIER SERIES AND INTEGRALS

Certain electrical and mechanical problems which lead to linear differential equations were studied in Secs. 11 to 13. In particular, a method of finding the steady-state response to a single sine term was explained. And the response to any linear combination of sine terms is the same linear combination of their responses. In this chapter we shall show how any periodic driving force may be represented exactly by an infinite series of sine terms, its Fourier series, or approximately by a finite sum of sine terms, a harmonic analysis. And many nonrecurrent driving forces may be represented by an integral involving sines, the Fourier integral.

The interpretation of the Fourier series for functions known over a finite range only is explained with a view to the applications to boundary value problems in partial differential equations given in Chap. 4. And we discuss a modification of the Fourier integral, the Laplace transform, upon which we shall base our treatment of Heaviside's operational calculus in Chap. 5.

15. Average. Root Mean Square

Let x be a real variable, and $y = g(x)$ be any function given in the interval a,b. Then $\bar{y}$, the *average* of y with respect to x or the interval a,b is defined by the equation

$$\bar{y} = \frac{1}{b-a} \int_a^b y \, dx = \frac{1}{b-a} \int_a^b f(x) dx. \tag{1}$$

This definition makes

$$(b-a)\bar{y} = \int_a^b y \, dx. \tag{2}$$

hen a $< b$ and $\bar{y} > 0$, the left member is the area of a rectangle
base b-a and height $\bar{y}$, while the right member is the area
ounded by the graph of $y = f(x)$, the x axis, and the ordinates
$= a, x = b$. Thus we may define $\bar{y}$ geometrically as the height
a rectangle between these ordinates whose area equals that
nder the curve. This leads to a graphic method of estimating
verages, for we need only slide a transparent straightedge over
ie graph parallel to the x axis until the total area between it
nd the curve above the straightedge appears to be equal to that
etween it and the curve below the straightedge. This holds
hether $\bar{y}$ is positive, as in Fig. 20, or negative, as in Fig. 21.
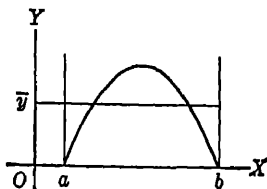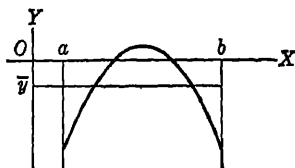
FIG. 20.   FIG. 21.

The geometric interpretation of Eq. (2) makes it easy to remem-
ber the definition.

As a particular example, consider the average of $y = 10x^m$,
$m > 0$, with respect to x on the interval 0,1. Since

$$\int_0^1 10x^m \, dx = \frac{10}{m+1} x^{m+1}\Big|_0^1 = \frac{10}{m+1}, \tag{3}$$

it follows that in this case

$$\bar{y} = \frac{1}{1-0} \cdot \frac{10}{m+1} = \frac{10}{m+1}. \tag{4}$$

For all values of m, $10x^m$ increases from 0 to 10 as x increases from
0 to 1. And if we take special values of m,

$$\text{For } m = 999 \quad 4 \quad 1 \quad 0.25 \quad \frac{1}{999}$$
$$\bar{y} = 0.01 \quad 2 \quad 5 \quad 8 \quad 9.99$$

This illustrates that the average always lies between the extreme values of $f(x)$ on a,b, here 0 and 10, but may be quite close to either. Again let y, u, and v be physical quantities related in such a way that

$$y = 10u^{1/999}, \qquad u = v^{(999)^2} \qquad \text{and hence} \qquad y = 10v^{999}, \quad (5)$$

Then if y increased from 0 to 10, u and v would each increase from 0 to 1. And, by the calculation just made, for the interval 0,1 the average of y with respect to u would be 9.99 while the average of y with respect to v would be 0.01. Thus for a physical quantity the average depends on the variable to which we refer it. In most applications this is the time.

The average $\bar{y}$ has the same units as y, and changes like y for a change of scale or units. The average does not depend on the scale or units of x, since a change in these would multiply $b - a$ and dx in Eq. (1) by the same factor.

We define $\bar{\bar{y}}$, the *root mean square* (rms) value of y with respect to x for the interval a,b, by the equation

$$\bar{\bar{y}} = \text{rms } y = \sqrt{\frac{\int_a^b y^2\, dx}{b - a}} \qquad (6)$$

Thus the rms y is the square root of the average of the square of y, and is so defined that

$$(b - a)(\bar{\bar{y}})^2 = \int_a^b y^2\, dx. \qquad (7)$$

This differs from Eq. (2) only in having $(\bar{\bar{y}})^2$ in place of $\bar{y}$, and y^2 in place of y.

Let us again take $y = 10x^m$, $m > 0$, on the interval 0,1. Then

$$\int_0^1 (10x^m)^2\, dx = \frac{100}{2m + 1}\, x^{2m+1}\Big|_0^1 = \frac{100}{2m + 1}, \qquad (8)$$

so that in this case

$$\bar{\bar{y}} = \sqrt{\frac{1}{1 - 0}\frac{100}{2m + 1}} = \frac{10}{\sqrt{2m + 1}} \qquad (9)$$

And if we take special values of m,

For m	999	4	1	0.25	$\frac{1}{999}$
$\bar{\bar{y}}$	0.224	3.333	5.773	8.166	9.99001

This illustrates that the rms value of y always lies between the extreme values of $f(x)$ on a,b, here 0 and 10, but may lie quite close to either. Unless y is constant, $\bar{\bar{y}}$ exceeds $\bar{y}$ as shown in Prob. 30 of Exercise VI. Our tables illustrate this, as do Eqs. (4) and (9) since $(m + 1)^2$ exceeds $2m + 1$.

16. Even Function. Odd Function

The polynomial

$$f(x) = 16 - 8x^2 + x^4 \tag{10}$$

has terms in x to the power 0, 2, 4 all even numbers. Hence its value is unchanged when we replace x by $-x$. That is,

$$f(-x) = f(x). \tag{11}$$

Its graph, Fig. 22, has the y axis as an axis of symmetry. Any function $f(x)$ for which Eq. (11) holds is said to be *even*. We may detect that a function is *even* from the symmetry of its graph about OY, by substituting in Eq. (11), or by observing that it is a combination of a finite or infinite number of terms, each of which has x to an even power. Any of these methods shows that cos x, sec x, cosh x, and sech x are all even functions. If the graph of an even function is known for values on one side of OY, the other half of the graph may

Fig. 22.   An even function.

be obtained by a reflection in OY. For example, $|x|$ is an even function equal to x for positive values of x. Hence its complete graph is as shown in Fig. 23.
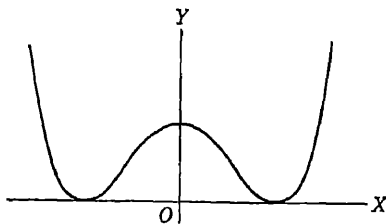
For $f(x)$ even, it follows from the symmetry of its graph that

$$\int_{-a}^{0} f(x)dx = \int_{0}^{a} f(x)dx = \tfrac{1}{2}\int_{-a}^{a} f(x)dx. \qquad (12)$$

$y = |x|$
FIG. 23.
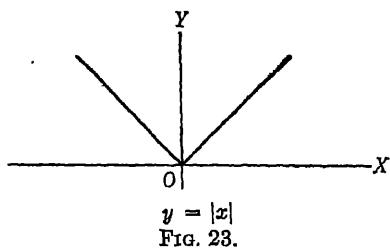
Dividing each of these by a converts the terms to averages. Hence *the average of an even function is the same for any one of the three intervals* $-a,0$; $0,a$; $-a,a$. This is also a direct consequence of the symmetry and the graphic method of finding averages.

Let us next consider the polynomial

$$f(x) = 4x + 4x^3 + x^5 \qquad (13)$$

which has terms in x to the power 1, 3, 5 all odd numbers. Hence it changes sign when we replace x by $-x$. That is,

$$f(-x) = -f(x). \qquad (14)$$

Its graph, Fig. 24, has such skew symmetry that for each point P of the graph there is a second point P' in the opposite quadrant for which the chord PP' is bisected by O. Any function $f(x)$ for which Eq. (14) holds is said to be *odd*. We may detect that a function is *odd* from the skew symmetry of its graph about O by substituting in Eq. (14) or by writing it as x times some combination of even powers. In particular the sum of a number of odd powers, or the re-

FIG. 24. An odd function.

procal of such a sum, is an odd function. But this is not necessarily true of other combinations of odd powers, for
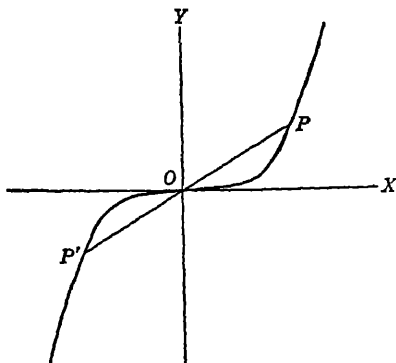
example, $(x^3)(x^5)$ which is even and $(x^3)(x^5) + x^7$ which is neither odd nor even. As examples of odd functions other than polynomials in odd powers we may mention $\sin x$, $\csc x$, $\tan x$, $\cot x$, $\sinh x$, $\operatorname{csch} x$, $\tanh x$, and $\coth x$. If the graph of an odd function is known for values on one side of OY, the other half of the graph may be obtained from it by a 180° rotation about O in the xy plane. For example, $(|x| - x^2)/x$, undefined for $x = 0$, is an odd function equal to $1 - x$ for positive values of x. Hence its complete graph is as shown in Fig. 25. As x increases through zero from negative to positive values, this $f(x)$ jumps from $f(0-) = -1$, the

FIG. 25. $y = (|x| - x^2)/x$.

value approached from the left, to $f(0+) = 1$, the value approached from the right. If we put $x = 0$ in Eq. (14), we find
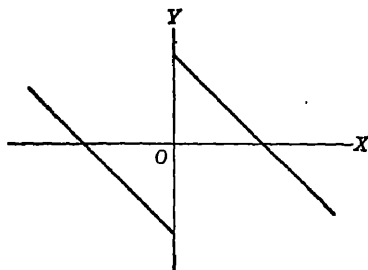
$$f(0) = -f(0), \quad \text{or} \quad 2f(0) = 0, \quad \text{and} \quad f(0) = 0. \quad (15)$$

Hence if we wish our function to be odd and defined for all x, including zero, we must put $f(0) = 0$, or add O to the graph.

For $f(x)$ odd, it follows from the nature of its graph that

$$\int_{-a}^{0} f(x)dx = -\int_{0}^{a} f(x)dx \quad \text{and} \quad \int_{-a}^{a} f(x)dx = 0. \quad (16)$$

Dividing each of these by a or $2a$ converts the terms into averages. Hence *the average of an odd function for the interval* $-a,0$ *is the negative of that for the interval* $0,a$ *and the average for the interval* $-a,a$ *is always zero.* This is also a direct consequence of the skew symmetry and the graphic method of finding averages.

The product of two even functions is even. The product of two odd functions is even. But the product of an odd and an even function is odd.

In particular the square of an even function is even, and the square of an odd function is even. Hence *for any even function,*

*or for any odd function, the root mean square value is the same for
any one of the three intervals* $-a,0; 0,a; -a,a$.

The general polynomial or series of powers will be neither odd
nor even.   However, it may be considered to be the sum of an
odd function, formed from the terms with odd powers, and an
even function, formed from the terms with even powers.   And
any function is the sum of an odd function and an even function,
since

$$g(x) = \frac{g(x) - g(x)}{2} + \frac{g(x) + g(-x)}{2}, \qquad (17)$$

and the first fraction changes sign, while the second is unchanged,
when we put $-x$ for x.   Starting with e^x which is neither odd
nor even, either the series or Eq. (17) would lead to

$$e^x = \sinh x + \cosh x,$$

and thus to $\sinh x$ as the odd component and $\cosh x$ as the even
component of e^x.

EXERCISE VI

Find the average of each of the following functions for the
interval 1,3:

1. $2 - 4x$.   **2.** $6x^2 + 8x^3$.   **3.** $2e^{-x}$.   **4.** $5 \ln x$.

Find the rms value of each of the following functions for the
interval 2,6:

5. $4 - x$.   **6.** $3e^{2x}$.   **7.** $\dfrac{1}{x-1}$.   **8.** $1 + x^2$.

Verify that each of the following functions is even and so has
the same average value for the intervals $-2,0; 0,2; -2,2$.   Com-
pute this average.   Also compute the rms for these intervals.

9. 5.   **10.** $2x^2$.   **11.** $3x^4$.   **12.** $\cos x$.

Verify that each of the following functions is odd and thus has
its average value 0 for the interval $-2,2$.   Compute the average

for the interval 0,2. Also compute the common value of the rms for the intervals $-2,0$; $0,2$; $-2,2$.

13. $4x$.   **14.** $8x^3$.   **15.** $\sin x$.   **16.** $\sin x \cos x$.

By inspection, find the average value over the interval $-5,5$ of each of the following functions:

17. $\sin^3 2x$.   **18.** $\sin x \cos^5 x + 4$.
19. $\sin 2x \cos^2 x + 2$.   **20.** $2x \cos 4x - 5$.

21. Find the average and rms value over the interval 0,4 of a function which is $2x$ when $0 < x < 2$ and is $8 - 2x$ when $2 < x < 4$ by the following methods: (*a*) by calculating the integral over 0,4 as that over 0,2 plus that over 2,4, and (*b*) by deducing from the graph that the average and rms for 0,4 is the same as that for 0,2.

By calculating the integral over $0, 2\pi/\omega$ as the sum of that over $0, \pi/\omega$ plus that over $\pi/\omega, 2\pi/\omega$ find for one cycle $0, 2\pi/\omega$ the average and rms value for

22. $E |\sin \omega t|$, the output of a full-wave rectifier.

23. $f(t) = E \sin \omega t$, when $0 < t < \pi/\omega$ and $f(t) = 0$ when $\pi/\omega < t < 2\pi/\omega$, the output of a half-wave rectifier.

24. If for y variable $du/dx = ky$, show that when x increases from a to b, u increases by the same amount as for $du/dx = k\bar{y}$, where $\bar{y}$ is the constant of Eq. (1).

25. From $F = m \, dv/dt$ and Prob. 24 deduce that in any interval t_1,t_2 the average force times the increase in time equals the gain in momentum, $\bar{F}(t_2 - t_1) = mv_2 - mv_1$.

26. In the definition of velocity, an "average" velocity $\dfrac{s_2 - s_1}{t_2 - t_2}$ is used. Show that this equals $\bar{v}$ for the interval t_1,t_2. Hint: Use $v = ds/dt$ and Prob. 24.

27. Let the interval a,b be divided into n equal parts by x_1, $x_2, \cdots, x_{n-1}$. As in Fig. 26, label the ordinates of the graph of $y = f(x)$ at these points $y_1, y_2, \cdots, y_{n-1}$, and that at b label y_n. The "average" of these n ordinates is

$$\bar{y}_n = (y_1 + y_2 + \cdots + y_n)/n.$$

Prove that as n becomes infinite, $\bar{y}_n \to \bar{y}$, the average defined by Eq. (1).

28. If for y variable $du/dx = ky^2$, show that when x increases from a to b, u increases by the same amount as for

$$du/dx = k(\bar{\bar{y}})^2,$$

where $\bar{\bar{y}}$ is the constant of Eq. (6).

Fig. 26.

29. If after t sec. H cal. of heat are generated by a current of i amperes flowing through a resistance of R ohms,

$$dH/dt = 0.24i^2R.$$

From this and Prob. 27 deduce that for any interval t_2, t_1 the calories generated $H_2 - H_1 = 0.24(\bar{i})^2R$.

30. For current i amperes the power P watts transferred to a load causing a drop in emf of e volts is $P = ei$. If the load has reactance $X = 0$, Sec. 12, $e = iR$, show that $P = \bar{e} \cdot \bar{i}$. But whenever the ratio e/i is not constant throughout the interval t_1, t_2 show that for this interval $P < \bar{e} \cdot \bar{i}$. HINT: Consider

$$\frac{1}{t_2 - t_1} \int_{t_1}^{t_2} (ix - e)^2 dt = \bar{i}^2 x^2 - 2\bar{P}x + \bar{e}^2,$$

where x is any real constant. Since the integral is positive, the same is true of the quadratic expression in x. Hence when equated to zero, the solution for x cannot be real, and the quantity under the radical $(-2\bar{P})^2 - 4(\bar{i}^2)(\bar{e}^2)$ is negative.

31. Prove that unless y is constant in a, b, $\bar{\bar{y}} > \bar{y}$. HINT: If $i = 1$ in Prob. 30, $i = 1$, $P = e$ and $e/i = e$. Hence when e is not constant in t_1, t_2, $\bar{e} < \bar{\bar{e}}$.

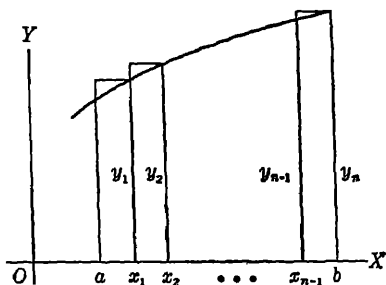If $y = c_1 f_1(x) + c_2 f_2(x)$, and $g = f_1 f_2$, show that for a, b.

32. The average $\bar{y} = c_1\bar{f_1} + c_2\bar{f_2}$.

33. The rms $\bar{\bar{y}} = \sqrt{c_1\bar{\bar{f_1}}^2 + 2c_1c_2\bar{g} + c_2\bar{\bar{f_2}}^2}$.

If the subscripts denote the interval over which the average or rms value of $y = f(x)$ is taken, show that

34. $\bar{y}_{ac} = \dfrac{(b-a)\bar{y}_{ab} + (c-b)\bar{y}_{bc}}{(c-a)}$.

35. $\bar{\bar{y}}_{ac} = \sqrt{\dfrac{(b-a)\bar{\bar{y}}_{ab}^2 + (c-b)\bar{\bar{y}}_{bc}^2}{(c-a)}}$.

A rod of variable density has length L and total mass M. It covers the interval $0,L$ of the x axis. Let $m = f(x)$ be the amount of mass on the interval $0,x$ and $x = f^{-1}(m)$ be the inverse function so that $f^{-1}(0) = 0$ and $f^{-1}(M) = L$. Show that

36. The distance of the center of gravity of the rod from 0 is $\bar{x}$, the average of $x = f^{-1}(m)$ on the interval $0,M$.

37. The radius of gyration of the rod about the y axis is $\bar{\bar{x}}$, the rms value of $x = f^{-1}(m)$ on the interval $0,M$.

17. Averages of Periodic Functions

A function $f(x)$ is said to be *periodic*, of period p, if

$$f(x + p) = f(x). \tag{18}$$

The graph of every function with period p, like that of Fig. 27, consists of a series of identical pieces such as those for the intervals $-p,0$; $0,p$; $p,2p$. It follows from Eq. (18) that

$$f(x + np) = f(x), \qquad n = 1, 2, \cdots \text{ or } -1, -2, \cdots . \tag{19}$$

Hence $f(x)$ is necessarily also of period np.

For any number a, the interval $a,a+p$ includes a point $b = 0$ or np. And the parts of the graph for a,b and $b,a+p$ are identical with the parts for c,p and $0,c$, where $c = a + p - b$.
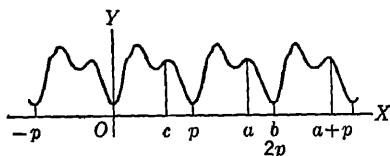

FIG. 27. A periodic function.

Hence we have

$$\int_a^{a+p} f(x)dx = \int_0^p f(x)dx. \tag{20}$$

By dividing the intervals $0, np$ or $a, a + np$, where $n > 0$, $p > 0$, into n equal parts each of length p, and applying Eq. (20) to each part, we find

$$\frac{1}{n} \int_0^{np} f(x)dx = \frac{1}{n} \int_a^{a+np} f(x)dx = \int_0^p f(x)dx. \tag{21}$$

Dividing by p converts each term of Eqs. (20) and (21) into an average, and proves that: *The average of a function of period p for any interval of length p or np is the same as that for the interval $0, p$.*

As shown in Prob. 39 of Exercise VII, for any interval which is very large as compared with the smallest period we get nearly the same value as for the interval $0, p$.

The periodic functions met in applications such as mechanical vibrations, alternating currents, or emfs are often functions of the time. And they are frequently known either exactly or approximately as sums of sine or cosine terms. In calculating rms values, we average e^2 and i^2, while for the average power we form the product ei and average. Thus we are led to averages of squares and products of sine and cosine terms. There are a few simple rules for calculating these special averages which we proceed to derive.

By Sec. 2, the complex function of the real variable x

$$e^{i(\omega x+\phi)} = \cos(\omega x + \phi) + i \sin(\omega x + \phi). \tag{22}$$

The phase ϕ may have any value, but we assume that the frequency $\omega > 0$. Like its real and imaginary parts, the complex function is of period $p = 2\pi/\omega$. This could be verified directly from the exponential form by using

$$e^{i\omega p} = e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1. \tag{23}$$

For, on putting $x + p$ for x in the exponential, we find

$$e^{i[\omega(x+p)+\phi]} = e^{i(\omega x+\phi)} \cdot e^{i\omega p} = e^{i(\omega x+\phi)}, \tag{24}$$

so that Eq. (18) is satisfied. Again, the integral

$$\int_0^p e^{i(\omega x + \phi)} dx = \frac{1}{\omega i} e^{i(\omega x + \phi)}\Big|_0^p = 0, \tag{25}$$

since by Sec. 3 the expression for the indefinite integral, obtained by the rule for real exponentials, has the integrand as its derivative and assumes the same value at the two limits by Eq. (24). Dividing by p converts the integral into an average, with real and imaginary parts averages of cosine and sine terms by Eq. (22). This proves our first rule:

I. *If* $\omega > 0$, *and* $p = 2\pi/\omega$, *the average of* $\cos(\omega x + \phi)$ *or* $\sin(\omega x + \phi)$ *for the interval* $0,p$ *is zero.* Since the functions are all of period p, the average is zero for any interval of length p or np with n a positive integer.

Let us next consider two positive frequencies ω_1, ω_2 whose ratio is a rational fraction n_1/n_2, the quotient of two positive integers. Then

$$\frac{\omega_1}{\omega_2} = \frac{n_1}{n_2}, \quad \text{so that} \quad \omega = \frac{\omega_1}{n_1} = \frac{\omega_2}{n_2}, \quad \omega_1 = n_1\omega, \quad \omega_2 = n_2\omega. \tag{26}$$

where ω is defined as the common value of the two equal fractions. The periods for frequencies ω, ω_1, ω_2 are

$$p = \frac{2\pi}{\omega}, \quad p_1 = \frac{2\pi}{\omega_1}, \quad p_2 = \frac{2\pi}{\omega_2}, \quad \text{and} \quad p = n_1 p_1 = n_2 p_2. \tag{27}$$

The product $\sin(\omega_1 x + \phi_1) \cos(\omega_2 + \phi_2)$ is necessarily periodic, of period p, since the first factor has p_1 and hence $n_1 p_1$ as a period while the second has p_2 and hence $n_2 p_2$. To find the average of the product for $0,p$, we first express each of the factors in terms of complex exponentials. By Sec. 2,

$$\sin(\omega_1 x + \phi_1) = \frac{e^{i(\omega_1 x + \phi_1)} - e^{-i(\omega_1 x + \phi_1)}}{2i}, \tag{28}$$

$$\cos(\omega_2 x + \phi_2) = \frac{e^{i\,\omega_2 x + \phi_2} + e^{-i(\omega_2 x + \phi_2)}}{2}. \tag{29}$$

The product will expand into four terms having in the exponent

as coefficients of ix either $\omega_1 + \omega_2$, $\omega_1 - \omega_2$ or the negatives of these quantities. Assume that $\omega_1 > \omega_2$, and decompose each of the exponentials by Sec. 2 into real and imaginary parts. This can lead only to a combination of sine and cosine terms with frequencies $\omega_3 = \omega_1 + \omega_2$ or $\omega_4 = \omega_1 - \omega_2$. The corresponding periods are

$$p_3 = \frac{2\pi}{\omega_1 + \omega_2} = \frac{p}{n_1 + n_2}, \qquad p_4 = \frac{2\pi}{\omega_1 - \omega_2} = \frac{p}{n_1 - n_2}, \quad (30)$$

where the second form is found by using the last two equalities of Eq. (26), and the first relation of Eq. (27). Since n_1 and n_2 are positive integers, and $\omega_1 > \omega_2$ implies $n_1 > n_2$, $n_3 = n_1 + n_2$ and $n_4 = n_1 - n_2$ are each positive integers. The averages of the sine and cosine terms with frequency ω_3, are zero for the interval $0,p_3$ by rule I, or for $0,n_3p_3$ and hence for $0,p$, since $p = n_3p_3$ by Eq. (30). Similarly for the terms with frequency ω_4 for $0,p$, since $p = n_4p_4$ by Eq. (30). And the argument just given leads to the same conclusion if $\omega_2 < \omega_1$ or if we replace the sine in the product by a cosine, or the cosine by a sine. Thus we have proved the second rule:

II. *Let* $\omega_1 > 0$, $p_1 = 2\pi/\omega_1$, $\omega_2 > 0$, $p_2 = 2\pi/\omega_2$ *and*

$$p = n_1p_1 = n_2p_2.$$

Then if $\omega_1 \neq \omega_2$, *the average of the product of the sine or cosine of* $(\omega_1 x + \phi_1)$ *times the sine or cosine of* $(\omega_2 x + \phi_2)$ *for the interval* $0,p$ *is zero.* Since each of these products is of period p, the average is zero for any interval of length p or np with n a positive integer.

If the frequencies ω_1 and ω_2 are equal, we may omit the subscripts and write the product $\sin (\omega x + \phi_1) \cos (\omega x + \phi_2)$. The previous argument still holds for the terms with frequency $\omega_1 + \omega_2 = 2\omega$, but the terms which were of frequency $\omega_1 - \omega_2$ are now the constant

$$\frac{e^{i(\phi_1 - \phi_2)} - e^{-i(\phi_1 - \phi_2)}}{2 \cdot 2i} = \frac{1}{2} \sin (\phi_1 - \phi_2), \qquad (31)$$

where the left member is obtained by putting $\omega_1 = \omega$ and $\omega_2 = \omega$ in Eqs. (28) and (29) and picking out the two exponential terms with a zero coefficient for x, while the reduction to the right member follows from Sec. 2. Since the average of a constant is the constant itself, this proves the third rule:

III. *If $\omega > 0$, and $p = 2\pi/\omega$, the average of the product*

$$\sin (\omega x + \phi_1) \cos (\omega x + \phi_2)$$

for the interval 0,p is equal to

$$\tfrac{1}{2} \sin (\phi_1 - \phi_2).$$

Since the product is of period p, the average has this same value for any interval of length p or np.

For the product $\sin (\omega x + \phi_1) \sin (\omega x + \phi_2)$, we use Eq. (28) with $\omega_1 = \omega$ and then with ϕ_1 replaced by ϕ_2 as well. The terms of the product constant in this case are equal to

$$\frac{e^{i(\phi_1-\phi_2)} + e^{-i(\phi_1-\phi_2)}}{2 \cdot 2} = \frac{1}{2} \cos (\phi_1 - \phi_2). \tag{32}$$

And this same constant is obtained for the product of two cosines $\cos (\omega x + \phi_1) \cos (\omega x + \phi_2)$ by use of Eq. (29). This proves the fourth rule:

IV. *If $\omega > 0$, and $p = 2\pi/\omega$, the average of the product*

$$\sin (\omega x + \phi_1) \sin (\omega x + \phi_2)$$

or

$$\cos (\omega x + \phi_1) \cos (\omega x + \phi_2)$$

for the interval 0,p is equal to $\tfrac{1}{2} \cos (\phi_1 - \phi_2)$. Since each of these products is of period p, the average has this same value for any interval of length p or np.

When $\phi_2 = \phi_1$, or in particular $\phi_1 = 0$ and $\phi_2 = 0$, the value given by rule IV is $\tfrac{1}{2} \cos 0 = \tfrac{1}{2}$. Hence we may state the fifth rule:

V. *If $\omega > 0$, and $p = 2\pi/\omega$, the average of the square*

$$\sin^2 (\omega x + \phi), \cos^2 (\omega x + \phi), \sin^2 \omega x, \text{ or } \cos^2 \omega x$$

for the interval $0,p$ *is equal to* $\frac{1}{2}$.

When $\phi_1 = 0$ and $\phi_2 = 0$, the value given by rule III is $\frac{1}{2} \sin 0 = 0$. We may combine this with rule II to give the sixth rule:

VI. *Let* $\omega_1 > 0$, $p_1 = 2\pi/\omega_1$, $\omega_2 > 0$, $p_2 = 2\pi/\omega_2$ *and*

$$p = n_1 p_1 = n_2 p_2.$$

Then if the product of $\sin \omega_1 x$ *or* $\cos \omega_1 x$ *times* $\sin \omega_2 x$ *or* $\cos \omega_2 x$ *is not a square, its average for the interval* $0,p$ *is zero.* For the square, both factors sines or both cosines and $\omega_2 = \omega_1$, the average is $\frac{1}{2}$ by rule V.

To illustrate the use of these rules, suppose that a load has such impedance that an impressed emf of e volts, where

$$e = 150.9 \sin 120\pi t - 31.33 \sin 360\pi t \tag{33}$$

causes a current of i amperes to flow through the load, where

$$i = 14 \sin (120\pi t - 21.9°) - 2 \sin (360\pi t - 50.33°). \tag{34}$$

For frequency $\omega = 120\pi$, $p = 2\pi/\omega = \frac{1}{60}$. And for frequency $\omega_1 = 360 = 3\omega$, $p_1 = 2\pi/(3\omega) = p/3$. Hence $P = \frac{1}{60} = 3p_1$ is a period for each term separately and therefore for the sums, so that we are dealing here with 60-cycle alternating current.

The power transferred to the load is $P = ei$. Let us calculate $\bar{P}$, the average value of P for the interval $0,\frac{1}{60}$. Using Eqs. (33) and (34), with ω for 120π, we may write

$$P = ei = 150.9 \times 14 \sin \omega t \sin (\omega t - 21.9°)$$
$$+ (-31.33)(-2) \sin 3\omega t \sin (3\omega t - 50.33°) + \cdots \tag{35}$$

where the dots stand for products of terms of different frequency ω, 3ω whose average is zero by rule II. For the terms written, the average is found by rule IV. Hence

$$\bar{P} = 150.9 \times 14 \times \frac{1}{2} \cos 21.9° + 31.33 \times 2 \times \frac{1}{2} \cos 50.33°$$
$$= 980 + 20 = 1,000 \text{ watts.} \tag{36}$$

If the rms value of i were required, we would write

$$i^2 = 14^2 \sin^2 (\omega t - 21.9°)$$
$$+ (-2)^2 \sin^2 (3\omega t - 50.33°) + \cdots, \quad (37)$$

using rule II for the omitted term, and find by rule V that

$$\overline{i^2} = (\bar{i})^2 = 14^2 \times \tfrac{1}{2} + 2^2 = 98 + 2 = 100. \quad (38)$$

Hence $\bar{i} = 100^{\frac{1}{2}} = 10$ amperes. We note that a similar calculation for e gives $\bar{e} = 154.1$ volts, so that $\bar{e} \cdot \bar{i} = 1{,}541$ exceeds $\bar{P}$. By Prob. 29 of Exercise VI, $\bar{e} \cdot \bar{i}$ must exceed P since the ratio of e/i changes with the time. The values just found for the interval $0, \tfrac{1}{60}$ are exact for any interval which is an integral number of cycles, for example, 5 min. which equals $18{,}000p$. Since any time interval larger than 5 min. is large compared with p, the averages for such an interval will be very close to those found for $0, \tfrac{1}{60}$ (see Prob. 40 of Exercise VII).

It is sometimes convenient to reduce all the terms to terms of zero phase. Thus if we expand the sines in Eq. (34) by the addition theorem, Eq. (11) of Sec. 2, it becomes

$$i = 12.990 \sin 120\omega t - 5.222 \cos 120\omega t - 1.277 \sin 360\omega t$$
$$+ 1.540 \cos 360\omega t. \quad (39)$$

From this and Eq. (33), using rules VI and V, we could find $\bar{P}$

$$P = 150.9 \times 12.990 \times \tfrac{1}{2} + (-31.33)(-1.277) \times \tfrac{1}{2}$$
$$= 980 + 20 = 1{,}000 \text{ watts.} \quad (40)$$

EXERCISE VII

1. Show that the function $f(x) = c$, a constant, is periodic, of period p, for any value of p.

Show that each of the following functions is periodic, and find the smallest possible value of the period p:

2. $7 \sin 5\pi x + 6 \sin 10\pi x + 2 \sin 15\pi x + 2 \sin 20\pi x$.

3. $24 - 13 \cos 20\pi x + 11 \cos 80\pi x - \cos 120\pi x$.

4. $8 \sin 12\pi x + 5 \sin 16\pi x$.   **5.** $6 \cos 12\pi x - 4 \cos 15\pi x$.

6. $10 \sin \dfrac{\pi x}{3} + 7 \sin \dfrac{\pi x}{2}$.   **7.** $9 \cos \dfrac{\pi x}{3} - 4 \cos \dfrac{\pi x}{5}$.

If $f(x) = x^2$ when $0 < x < 1$, and $f(x) = 2 - x$ when $1 < x < 2$, sketch the graph of $y = f(x)$ for $-6 < x < 6$ if

8. $f(x)$ is periodic, of period 2.

9. $f(x)$ is an even function and periodic, of period 4.

10. $f(x)$ is an odd function and periodic, of period 4.

11. If $f(x) = 4x - 4x^2$ when $0 < x < 1$, and $f(x)$ is periodic of period 1, show that $f(x)$ is an even function.

12. If $f(x) = x^3 - 3x^2 + 2x$ when $0 < x < 2$, and $f(x)$ is periodic of period 2, show that $f(x)$ is an odd function.

In each case, find the rms value for an interval equal to a complete period of the sum:

13. $e = 200 \cos 120\pi t - 100 \sin 120\pi t + 30 \sin 360\pi t$.

14. $i = 160 \cos 120\pi t - 80 \sin 120\pi t + 24 \sin 360\pi t$.

15. $e = 240 \cos 60\pi t + 48 \cos 180\pi t$.

16. $i = 2.74 \cos (60\pi t - 31°) + 0.42 \cos (180\pi t - 61°)$.

17. $e = 250 \sin (50\pi t + 78.67°) + 50 \sin (150\pi t - 2.83°)$.

18. $i = 20 \sin 50\pi t + 3 \cos 150\pi t$.

19. $e = 400 \cos (120\pi t + 50°) - 70 \sin (360\pi t + 42°)$.

20. $i = 2.56 \sin (120\pi t + 88.83°) - 0.187 \sin (360\pi t - 33.5°)$.

21. $e = 25 \sin \left(\dfrac{\pi t}{30} + 75° \right) + 5 \sin \left(\dfrac{\pi t}{10} - 75° \right)$.

22. $i = 25 \sin \left(\dfrac{\pi t}{30} + 73.17° \right) + 5 \sin \left(\dfrac{\pi t}{10} - 80.67° \right)$.

Find the average power for a complete period transferred by a current of i amperes due to an emf of e volts given in

23. Probs. 13 and 14.   **24.** Probs. 15 and 16.

25. Probs. 17 and 18.   **26.** Probs. 19 and 20.

27. Probs. 21 and 22.

28. Show that 1 sec. is a possible period for each of the sums given in Probs. 13 to 20, and find the number of cycles per second in each case.

29. If $i = I_1 \sin (\omega t + a_1) + I_3 \sin (3\omega t + a_3)$ and $e = E_1 \sin (\omega t + b_1) + E_3 \sin (3\omega t + b_3)$, show that for a complete

le the average value of the power P is

$$\bar{P} = \tfrac{1}{2}[E_1 I_1 \cos (a_1 - b_1) + E_3 I_3 \cos (a_3 - b_3)]$$

d that the rms values are $\bar{\imath} = \sqrt{\tfrac{1}{2}(I_1{}^2 + I_3{}^2)}$ and

$$\bar{e} = \sqrt{\tfrac{1}{2}(E_1{}^2 + E_3{}^2)}.$$

le results are similar if there are additional terms each with bscript n and frequency $n\omega$, for n some positive integer.

30. Show that the results of Prob. 29 hold if in i and e we place the sines by cosines.

31. If $i = A_1 \cos \omega t + C_1 \sin \omega t + A_3 \cos 3\omega t + C_3 \sin 3\omega t$ id $e = B_1 \cos \omega t + D_1 \sin \omega t + B_3 \cos 3\omega t + D_3 \sin 3\omega t$, show iat for a complete cycle the average value of the power P is

$= \tfrac{1}{2}(A_1 B_1 + C_1 D_1 + A_3 B_3 + C_3 D_3)$ and that the rms values

:e $\bar{\imath} = \sqrt{\tfrac{1}{2}(A_1{}^2 + C_1{}^2 + A_3{}^2 + C_3{}^2)}$,

$$\bar{e} = \sqrt{\tfrac{1}{2}(B_1{}^2 + D_1{}^2 + B_3{}^2 + D_3{}^2)}.$$

'he results are similar if there are additional terms each with ubscript n and frequency $n\omega$, for n some positive integer.

32. Show that the equations $A_n + I_n \sin a_n, C_n = I_n \cos a_n$:re solved by $I_n = \sqrt{A_n{}^2 + C_n{}^2}$, $\tan a_n = \dfrac{A_n}{C_n}$ without canceling ninus signs as in Sec. 6. Assume that these equations as well as .hose with I, a, A, C replaced by E, b, B, D hold for each n :resent in the sums of Prob. 29 or 31. Deduce that this makes :he given i and e, as well as the conclusions identical. Such relations may be used to convert either given form into the other.

33. Show that the equations $A_n = I_n \cos a_n, C_n = -I_n \sin a_n$ are solved by $I_n = \sqrt{A_n{}^2 + C_n{}^2}$, $\tan a_n = -\dfrac{C_n}{A_n}$ without cancel-ing minus signs as in Sec. 6. Assume that these equations as well as those with I, a, A, C replaced by E, b, B, D hold for each n present in the sums of Probs. 30 or 31. Deduce that this makes the given i and e, as well as the conclusions, identical. Such relations may be used to convert either given form into the other.

34. Use Prob. 32 to convert Eq. (34) into Eq. (39).

35. Use Prob. 32 to convert Eq. (39) into Eq. (34).

36. Show that if in Eq. (26) n_1 and n_2 are both odd integers the products of rule II, or when $n_1 = n_2 = 1$ the squares of rule V, are of period π/ω, one-half the p used in the rules.

Verify the following illustrations of Prob. 36:

37. $\sin 3\pi x \cos 5\pi x$, $n_1 = 3$, $n_2 = 5$, $2\pi/\omega = 2$, but 1 is a period.

38. $\sin^2 4\pi x$, $2\pi/4 = \frac{1}{2}$, but $\frac{1}{4}$ is a period.

39. Let $f(x)$ satisfy Eq. (18), with $p > 0$. Then any number L large compared with p can be written $L = np + c$, where n is a large positive integer and $0 \leqq c < p$. Show that

$$L\bar{f}_L = np\bar{f}_p + c\bar{f}_c,$$

where the intervals for the averages are a, $a + L$ for $\bar{f}_L$, $0,p$ for $\bar{f}_p$ and $0,c$ for $\bar{f}_c$. Deduce that

$$\bar{f}_L - \bar{f}_p = \frac{np\bar{f}_p + c\bar{f}_c}{np + c} - \bar{f}_p = \frac{c(\bar{f}_c - \bar{f}_p)}{np + c} < \frac{f_c - \bar{f}_p}{n}$$

which is numerically less than $2M/n$, where M is the maximum or largest extreme value of $|f(x)|$. This proves that for any interval of length L, so large compared with p that its ratio to p is large compared with the values of $|f(x)|$, the average $\bar{f}_L$ is nearly the same as $\bar{f}_p$ that for $0,p$.

40. In Prob. 40 take $f(x) = A \sin \omega x$, $p = 2\pi/\omega$ and $c = p/2$, or $L = (n + \frac{1}{2})p$. Verify that $f_c = 0.707A$, $\bar{f}_p = 0$, and $\bar{f}_L - \bar{f}_p = \dfrac{0.707A}{2n + 1}$. With x the time in seconds and $\omega = 120\pi$, $p = \frac{1}{60}$ sec. And $\bar{f}_L - \bar{f}_p = 0.006A$ for $n = 60$, or $0.00002A$ for $n = 18{,}000$, which are good indications of the possible departure of $\bar{f}_L$ from $\bar{f}_p = 0$ for times exceeding 1 sec. or 5 min.

18. Fourier's Theorem for Periodic Functions

If $\omega = 2\pi/p$, so that $p = 2\pi/\omega$, and n is any integer, the function

$$e^{in\omega x} = \cos n\omega x + i \sin n\omega x \tag{41}$$

is periodic of period p. For, on putting $x + p$ for x, we find

$$e^{in\omega(x+p)} = e^{in\omega x} \cdot (e^{i\omega p})^n = e^{in\omega x},$$

by Eq. (23), so that Eq. (18) is satisfied.   Hence any sum or series of terms each of which is a constant times $e^{in\omega x}$, its real part $\cos n\omega x$, or its imaginary part $\sin n\omega x$, will be of period p.   In particular, if the infinite series

$$a + a_1 \cos \omega x + b_1 \sin \omega x + a_2 \cos 2\omega x + b_2 \sin 2\omega x + \cdots$$
$$+ a_n \cos n\omega x + b_b \sin n\omega x + \cdots \quad (42)$$

or

$$a + \sum_{k=1}^{\infty} (a_b \cos k\omega x + b_k \sin k\omega x) \quad (43)$$

is convergent, it represents a periodic function of period p where

$$p = \frac{2\pi}{\omega} \qquad \text{or} \qquad \omega = \frac{2\pi}{p}. \quad (44)$$

If a function $f(x)$ is single-valued and continuous on a finite interval and its graph on this interval has finite arc length, we call the function or its graph *regular*.   We call a single-valued function $f(x)$ *piecewise regular* if its graph on any finite interval is made up of a finite number of pieces, each of which is a regular arc or an isolated point.

For example, on the interval $0 \leqq x < 8$, the relations

$$f(0) = 1, \qquad f(x) = 2 \qquad \text{if } 0 < x < 2, \qquad f(2) = 1,$$
$$f(x) = 0 \qquad \text{if } 2 < x < 8, \quad (45)$$

define a piecewise regular function.   And, if we add the condition

$$f(x + 8) = f(x), \quad (46)$$

the function is defined for all values of x as a piecewise regular function of period 8.   Its graph is shown in Fig. 28.   We use the notation $f(x-)$ to mean the value at x approached from the left and $f(x+)$ to mean the value at x approached from the right. Thus at $x = 2$ these values are $f(2-) = 2$ and $f(2+) = 0$.   If x is the time with a suitable submultiple of a second as the unit, this $f(x)$ would represent a rectangular pulse, repeated at an interval four times its duration, of a type often used in testing receiving apparatus.

Suppose that $f(x)$ is any piecewise regular periodic function of period p. Then it may be proved that there are coefficients for which the series (43), with $\omega = 2\pi/p$, converges to $f(x)$ at all points of continuity, and to $\frac{1}{2}[f(x+) + f(x-)]$ at the points of discontinuity. This series is called the *Fourier series of the*
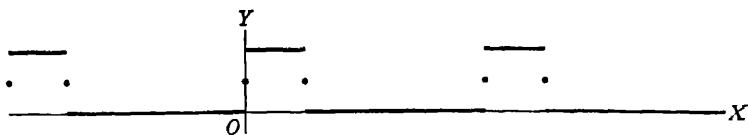

FIG. 28. A piecewise regular periodic function.

periodic function $f(x)$. Furthermore, correct relations will be obtained from the equation

$$f(x) = a + \sum_{k=1}^{\infty} (a_k \cos k\omega x + b_k \sin k\omega x), \qquad (47)$$

by termwise integration after multiplication by any function of x.

Let us use as multipliers 1, $\cos n\omega x$, $\sin n\omega x$, respectively, where n is any positive integer, and integrate from c to $c + p$, where c is any constant. Dividing by p converts the integrals into averages which may be found for the terms on the right by the rules of Sec. 17. For 1 we find

$$\frac{1}{p} \int_c^{c+p} f(x)dx = a, \qquad \text{or} \qquad a = \frac{1}{p} \int_c^{c+p} f(x)dx \qquad (48)$$

by rule I. For $\cos n\omega x$ and $\sin n\omega x$ we find by rules VI and V

$$\frac{1}{p} \int_c^{c+p} f(x) \cos n\omega x \, dx = \frac{a_n}{2}, \qquad \text{or}$$

$$a_n = \frac{2}{p} \int_c^{c+p} f(x) \cos n\omega x \, dx, \qquad (49)$$

$$\frac{1}{p} \int_c^{c+p} f(x) \sin n\omega x \, dx = \frac{b_n}{2}, \qquad \text{or}$$

$$b_n = \frac{2}{p} \int_c^{c+p} f(x) \sin n\omega x \, dx. \qquad (50)$$

To recapitulate, if $f(x)$ is any piecewise regular periodic function of period p; and a, the a_n, and the b_n are found from Eqs. (48) to (50), then the series (43), with $\omega = 2\pi/p$, will converge to $f(x)$ at all points of continuity, and to $\frac{1}{2}[f(x+) + f(x-)]$ at all points of discontinuity. This is known as *Fourier's theorem for periodic functions*.

Let us illustrate the procedure for the function defined by the relations (45) and (46). Here $p = 8$, $\omega = \pi/4$. We take $c = 0$ and, since the expression for the function changes at 2, calculate the integrals from 0 to 8 as the sum of those from 0 to 2 and from 2 to 8. Thus we find

$$a = \tfrac{1}{8} \left(\int_0^2 2dx + \int_2^8 0dx \right) = \tfrac{1}{8}(4 + 0) = \tfrac{1}{2}. \tag{51}$$

$$a_n = \frac{2}{8} \left(\int_0^2 2 \cos n\omega x \, dx + \int_2^8 0 \cos n\omega x \, dx \right)$$
$$= \frac{1}{4} \left(\frac{2}{n\omega} \sin n\omega x \Big|_0^2 + 0 \right) = \frac{\sin 2n\omega}{2n\omega} = \frac{2}{n\pi} \sin \frac{n\pi}{2}. \tag{52}$$

$$b_n = \frac{2}{8} \left(\int_0^2 2 \sin n\omega x \, dx + \int_2^8 0 \sin n\omega x \, dx \right)$$
$$= \frac{1}{4} \left(-\frac{2}{n\omega} \cos n\omega x \Big|_0^2 + 0 \right) = -\frac{\cos 2n\omega - 1}{2n\omega}$$
$$= \frac{2}{n\pi} \left(1 - \cos \frac{n\pi}{2} \right). \tag{53}$$

To save writing we carry the calculation as far as possible before replacing ω by its value, here $\pi/4$. Putting n successively equal to 1, 2, 3, etc., we find from Eqs. (51) to (53) that the Fourier series for the function $f(x)$ of (45) and (46) is

$$\frac{1}{2} + \frac{2}{\pi} \left(\cos \omega x - \frac{1}{3} \cos 3\omega x + \frac{1}{5} \cos 5\omega x - \cdots \right)$$
$$+ \frac{2}{\pi} \left(\sin \omega x + \frac{2}{2} \sin 2\omega x + \frac{1}{3} \sin 3\omega x + * + \frac{1}{5} \sin 5\omega x + \cdots \right) \tag{54}$$
$$\text{where } \omega = \pi/4.$$

Not all rearrangements of the terms of a Fourier series will give a convergent series, but it is always allowable to group the sine

tèrms together and the cosine terms together into two separate
series as we have done here.

The defining relations (45) and (46) happen to make the value
of the function equal to $\frac{1}{2}[f(x+) + f(x-)]$ at the points of dis-
continuity.   For   example, $f(8) = \frac{1}{2}[f(8+) + f(8-)]$,   since
$1 = \frac{1}{2}(2 + 0)$.   Hence Fig. 28 is the graph of the sum of the
series (54) for all values of x.   Had we taken other values for
$f(x)$ as the definition at the points of discontinuity, we would
still have found the same Fourier series since the values $f(0)$ and
$f(2)$ were not used in Eqs. (51) to (53).   We may always obtain
the graph of the Fourier series which represents any piecewise
regular function by plotting the regular arcs, together with the
mid-points of the vertical segments determined by consecutive
arcs.   In Fourier series problems, we shall frequently define the
regular arcs only and give no values at the discontinuities, since
these last do not affect the Fourier series.

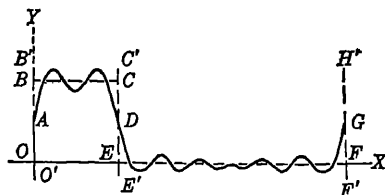The full curve of Fig. 29 is the graph of the sum of the terms

FIG. 29.

of the series (54), up to and including those involving $7\omega x$.   The
points A, D, G together with the interior points of the dotted
horizontal intervals BC and EF are on the graph of the sum of
the series.   Note that, while the partial sum is almost equal to
$f(x)$ for $x = 0$, 2, 8 where $f(x)$ has a discontinuity, and is a fair
approximation to $f(x)$ for x well inside the intervals 0,2 and 2,8,
the approximation is poor for values of x near but not at 0, 2, 8.
The distance from the minimum nearest 2 to the maximum near-
est 2 is greater than the jump of the function at 2.   This excess
persists for all the sums, so that if we plotted the sum for a very

large number of terms it would be indistinguishable from the set of dotted segments $O'B'$, BC, $C'E'$, EF, $F'H'$. This is typical of the behavior of sums of Fourier series near points of discontinuity. The sum for a large number of terms is always indistinguishable from the regular arcs, together with the vertical segments joining their ends extended by about 18 per cent. That is, to a segment like EC we add EE' and CC', each 9 per cent of EC. The overshooting of the jumps in this way is known as *Gibbs's phenomenon.*

EXERCISE VIII

Find the Fourier series of a function which is of period 2π and is defined in the interval $-\pi < x < \pi$ as equal to

1. x. **2.** x^2. **3.** x^3. **4.** e^x. **5.** $x \sin x$. **6.** $x \cos x$.

7. $f(x) = \pi$ if $-\pi < x < 0$ and $f(x) = 0$ if $0 < x < \pi$.
8. $f(x) = x$ if $-\pi < x < 0$ and $f(x) = 0$ if $0 < x < \pi$.

Find the Fourier series of a function which is of period 2π and is defined in the interval $0 < x < 2\pi$ as equal to

9. x. **10.** x^2. **11.** x^3 **12.** e^x. **13.** $x \sin x$. **14.** $x \cos x$.

15. $f(x) = \pi$ if $0 < x < \pi$ and $f(x) = 0$ if $\pi < x < 2\pi$.
16. $f(x) = x$ if $0 < x < \pi$ and $f(x) = 0$ if $\pi < x < 2\pi$.

Find the Fourier series of a function $f(x)$ which is of period 10, and such that $f(x) = 0$ if $-5 < x < 0$, and if $0 < x < 5$, $f(x)$ is equal to

17. 10. **18.** x. **19.** e^x. **20.** $\sin x$. **21.** $\cos x$.

22. Show that the Fourier series for a rectangular pulse of unit height, duration w, on from c to $c + w$ and repeated at intervals $2\pi/\omega$, Fig. 30, may be written

$$\frac{\omega w}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\omega w}{2} \cos n\omega \left(x - c - \frac{w}{2} \right).$$

23. Use Prob. 22 to check Probs. 7, 15, and 17 and Eq. (54).

24. Show that the Fourier series for a function of period $2\pi/\omega$ which equals x for $c < x < c + 2\pi/\omega$ may be written

$$x = c + \frac{\pi}{\omega} - \frac{2}{\omega} \sum_{n=1}^{\infty} \frac{1}{n} \sin n\omega(x - c), \quad c < x < c + \frac{2\pi}{\omega}.$$
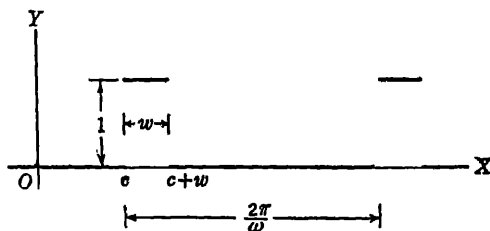

Fig. 30. A rectangular pulse.

25. Use Prob. 24 to check Probs. 1 and 9.

26. Show that the Fourier series for a function $f(x)$ of period $2\pi/\omega$ such that

$$f(x) = 0 \text{ if } c - \frac{\pi}{\omega} < x < c, \ f(x) = x \text{ if } c < x < c + \frac{\pi}{\omega},$$

may be written

$$\frac{c}{2} + \frac{\pi}{4\omega} + \left(\frac{2c}{\pi} + \frac{1}{\omega}\right) \sum_{n=1}^{\infty} \frac{\sin n\omega \, (x - c)}{n} - \frac{2}{\pi\omega} \sum_{n=1}^{\infty} \frac{\cos n\omega \, (x - c)}{n^2}.$$

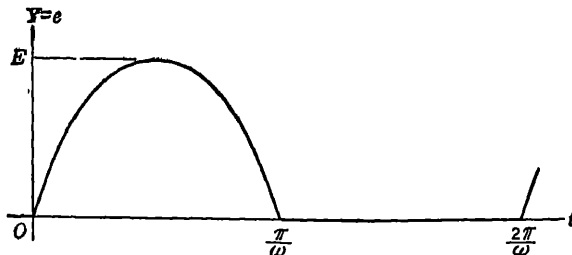**27.** Use Prob. 26 to check Probs. 8, 16, and 18.

Fig. 31. Output voltage of a half-wave rectifier.

28. When an emf $E \sin \omega t$ is impressed on a half-wave rectifier, the output voltage e, Fig. 31, of period $2\pi/\omega$, is zero from

$t = -\pi/\omega$ to $t = 0$ and $e = E \sin \omega t$ if $0 < t < \pi/\omega$. Show that

$$e = E \left(\frac{1}{\pi} + \frac{1}{2} \sin \omega t - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\omega t}{4n^2 - 1} \right)$$

$$= \frac{E}{\pi} (1 + 1.57 \sin \omega t - 0.67 \cos 2\omega t - 0.13 \cos 4\omega t$$

$$- 0.06 \cos 6\omega t - \cdots).$$

29. Use Prob. 28 to check Prob. 20.

19. Half-range Fourier Series

For *any* given function $f(x)$, Eqs. (48) to (50) and Eq. (44) may be used to find a *Fourier series of period* p, Eq. (43), which represents $f(x)$ for $c < x < c + p$, where $c, c + p$ is any interval on which $f(x)$ is piecewise regular. For, from $f(x)$ on $c, c + p$ we may form a function of period p equal to the given function for $c < x < c + p$. The series (43) and (44) with coefficients calculated by Eqs. (48) to (50) will then represent the periodic function for all x. Hence in particular it will represent the given $f(x)$ for $c < x < c + p$.

The interval $c, c + p$ reduces to $-L, L$ with center at the origin if $c = -L$ and $p = 2L$. With these values Eqs. (48) to (50) and Eq. (44) become

$$a = \frac{1}{2L} \int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos n\omega x \; dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin n\omega x \; dx, \qquad \omega = \frac{\pi}{L}. \qquad (55)$$

Let $f(x)$ be an even function as defined in Sec. 16. Then the integrands $f(x)$ and $f(x) \cos n\omega x$ are each even, while $f(x) \sin n\omega x$ is odd. From Eqs. (12), (16), and (55) we may conclude that $b_n = 0$ and that

$$a = \frac{1}{L} \int_0^L f(x)dx, \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos n\omega x \; dx. \qquad (56)$$

Hence with these values, for any *even* $f(x)$ of period $2L$, and all x,

$$f(x) = a + \sum_{k=1}^{\infty} a_k \cos n\omega x, \qquad \text{where } \omega = \frac{\pi}{L}. \qquad (57)$$

For *any* given function $f(x)$, piecewise regular on $0,L$, Eqs. (56) and (57) may be used to find a *Fourier cosine series of period* $2L$ which represents $f(x)$ in $0 < x < L$. For, from $f(x)$ on $0,L$ we may form an even function of period $2L$ equal to the given function for $0 < x < L$. The series (57) with coefficients calculated by Eq. (56) will then represent the even periodic function
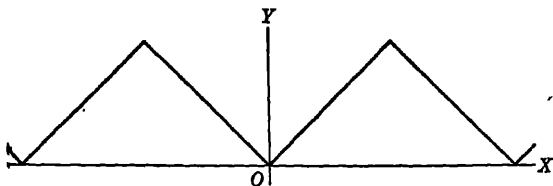

FIG. 32. An even function.

for all x. Hence in particular it will represent the given $f(x)$ for $0 < x < L$.

We illustrate the procedure by finding the Fourier cosine series of period 2π which represents x in the interval $0 < x < \pi$. On putting $f(x) = x$ and $L = \pi$, $\omega = 1$ in Eq. (56), we find

$$a = \frac{1}{\pi} \int_0^\pi x \, dx = \frac{1}{\pi} \frac{x^2}{2} \Big|_0^\pi = \frac{\pi}{2}. \qquad (58)$$

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos nx \, dx = \frac{2}{\pi} \left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right) \Big|_0^\pi$$
$$= \frac{2}{\pi n^2} (\cos n\pi - 1). \qquad (59)$$

It follows from these values and Eq. (57) that, for $0 < x < \pi$,

$$x = \frac{\pi}{2} - \frac{4}{\pi} \left(\cos x + \frac{1}{3^2} \cos 3x + \frac{1}{5^2} \cos 5x + \cdots \right). \qquad (60)$$

The graph of the even periodic function with this as its Fourier series is as shown in Fig. 32. Hence Eq. (60) holds for $x = 0$,

or $x = \pi$ but not for $x < 0$ or $x > \pi$. The full curve of Fig. 33 is the graph of the sum of the terms of the series (60), up to and including that in $5x$.   Comparison with the sum of the series, shown as a dotted curve, shows that we have a fair approximation at all points.

Next suppose that $f(x)$ is an odd function as defined in Sec. 16. Then in Eq. (55) the integrands $f(x)$ and $f(x) \cos n\omega x$ are each odd, so that by Eq. (16) $a = a_n = 0$.   But $f(x) \sin n\omega x$ is even,
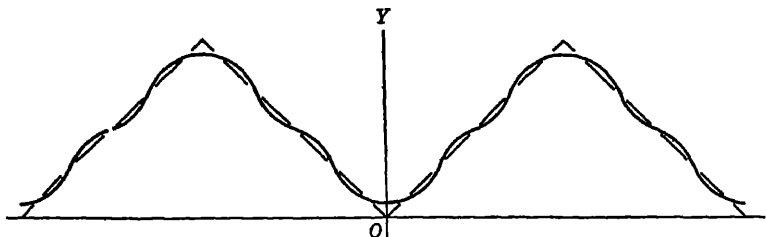

Fig. 33.   Partial sum of a Fourier series.

and we may conclude from Eqs. (55) and (12) that

$$b_n = \frac{2}{L} \int_0^L f(x) \sin n\omega x \, dx. \tag{61}$$

Hence with these values, for any *odd* $f(x)$ of period $2L$, and all x,

$$f(x) = \sum_{k=1}^{\infty} b_k \sin k\omega x, \qquad \text{where } \omega = \frac{\pi}{L}. \tag{62}$$

For *any* given function $f(x)$, piecewise regular on $0,L$, Eqs. (61) and (62) may be used to find a *Fourier sine series of period* $2L$ which represents $f(x)$ in $0 < x < L$.   For, from $f(x)$ on $0,L$ we may form an odd function of period $2L$ equal to the given function for $0 < x < L$.   The series (62) with coefficients calculated by Eq. (61) will then represent the odd periodic function for all x.   Hence in particular it will represent the given $f(x)$ for $0 < x < L$.

We illustrate the procedure by finding the Fourier sine series of period 2π which represents $\pi/2$ in the interval $0 < x < \pi$.

On putting $f(x) = \pi/2$ and $L = \pi$, $\omega = 1$ in Eq. (61), we find

$$b_n = \frac{2}{\pi} \int_0^\pi \frac{\pi}{2} \sin nx \, dx = -\frac{1}{n} \cos nx \Big|_0^\pi = \frac{1 - \cos n\pi}{n}. \quad (63)$$

It follows from these values and Eq. (62) that, for $0 < x < \pi$,

$$\frac{\pi}{2} = 2 \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right). \quad (64)$$

The graph of the odd periodic function with this as its Fourier series is as shown in Fig. 34. Hence Eq. (64) does not hold for
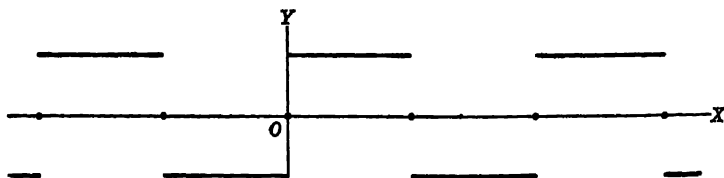
FIG. 34.   An odd function.

$x = 0$ or $x = \pi$.

For the period $p = 2L$, a function $f(x)$ is called *odd-harmonic* if

$$f(x + L) = -f(x). \quad (65)$$

Such a function is necessarily periodic, of period p, since

$$f(x + p) = f[(x + L) + L] = -f(x + L) = f(x). \quad (66)$$

FIG. 35.   An odd-harmonic function.

And as Fig. 35 illustrates for the graph of $f(x)$, the piece for any interval of length L can be obtained from the piece for the preceding interval of length L by reflecting in the x axis and advanc-

ing a distance L. It follows that

$$\int_{-L}^{0} f(x)dx = - \int_{0}^{L} f(x)dx \qquad \text{and} \qquad \int_{-L}^{L} f(x)dx = 0. \quad (67)$$

The oscillographs of many of the emfs generated in practice exhibit the odd-harmonic property.

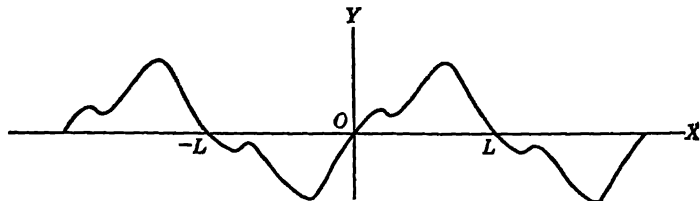Equation (65) suggests that we study the related property

$$f(x + L) = f(x). \tag{68}$$

Any function $f(x)$ for which Eq. (68) holds is said to be *even-harmonic* for the period $p = 2L$. Such a function is periodic, of period L, by Eq. (18), and hence of period $2L$ by Eq. (19). And from Eq. (21),

$$\frac{1}{2} \int_{-L}^{L} f(x)dx = \int_{0}^{L} f(x)dx. \tag{69}$$

Let $\omega = 2\pi/p = \pi/L$, so that $L = \pi/\omega$. Then

$$e^{i\omega L} = e^{i\pi} = \cos \pi + i \sin \pi = -1. \tag{70}$$

It follows from this that

$$e^{in\omega(x+L)} = e^{in\omega x} \cdot (e^{i\omega L})^n = (-1)^n e^{in\omega x} \tag{71}$$

$$= \left\{ \begin{array}{ll} e^{in\omega x}, & \text{if } n = 0, 2, 4, 6, \cdots . \quad (72) \\ -e^{in\omega x}, & \text{if } n = 1, 3, 5, \cdots . \quad (73) \end{array} \right.$$

This shows that $e^{in\omega x}$, and hence its real and imaginary parts $\cos n\omega x$ and $\sin n\omega x$ by Eq. (40), is even-harmonic for the period $2L$ when n is 0 or an even integer, but is odd-harmonic for the period $2L$ when n is an odd integer.

Now let $f(x)$ in Eq. (55) be odd-harmonic for the period $2L$. Since the product of an odd-harmonic function and an even-harmonic function is odd-harmonic, for n an even integer the integrands $f(x) \cos n\omega x$ and $f(x) \sin n\omega x$ are odd-harmonic. Hence by Eq. (57) the integrals in Eq. (55) are zero, and

$$a = 0, \quad a_2 = 0, \quad b_2 = 0, \quad a_4 = 0, \quad b_4 = 0, \quad \cdots . \tag{74}$$

But the product of two odd-harmonic functions is even-harmonic. Hence for n an odd integer, the integrands $f(x) \cos n\omega x$ and $f(x) \sin n\omega x$ are even-harmonic, and we may use Eq. (69) to

transform the integrals.   Thus

$$a_m = \frac{2}{L} \int_0^L f(x) \cos m\omega x \, dx, \qquad b_m = \frac{2}{L} \int_0^L f(x) \sin m\omega x \, dx,$$
$$m = 1, 3, 5, \cdots . \tag{75}$$

With these values, for any $f(x)$ which is odd-harmonic for the period $2L$, for all x,

$$f(x) = \Sigma(a_m \cos m\omega x + b_m \sin m\omega x), \tag{76}$$

where the sum is taken over all odd integers, $m = 1, 3, 5, \cdots$.

For *any* given function $f(x)$, piecewise regular on $0,L$, Eqs. (75) and (76) may be used to find an *odd-harmonic Fourier series of period $2L$* which represents $f(x)$ in $0 < x < L$.   For, from $f(x)$ on $0,L$ we may form a function which is odd-harmonic for the period $2L$ equal to the given function for $0 < x < L$.   The series (76) with coefficients calculated by Eq. (75) will then represent the odd-harmonic function for all x.   Hence in particular it will represent the given $f(x)$ for $0 < x < L$.

We illustrate the procedure by finding the odd-harmonic Fourier series of period 2π which represents x in the interval $0 < x < \pi$.   On putting $f(x) = x$ and $L = \pi$, $\omega = 1$ in Eq. (75), we find

$$a_m = \frac{2}{\pi} \int_0^\pi x \cos mx \, dx = \frac{2}{\pi} \left(\frac{x \sin mx}{m} + \frac{\cos mx}{m^2} \right) \Big|_0^\pi$$
$$= -\frac{4}{\pi m^2}. \tag{77}$$

$$b_m = \frac{2}{\pi} \int_0^\pi x \sin mx \, dx = \frac{2}{\pi} \left(-\frac{x \cos mx}{m} + \frac{\sin mx}{m^2} \right) \Big|_0^\pi = \frac{2}{m}. \tag{78}$$

It follows from these values and Eq. (76) that for $0 < x < \pi$,

$$x = -\frac{4}{\pi} \sum \frac{1}{m^2} \cos mx + 2 \sum \frac{1}{m} \sin mx,$$
$$m = 1, 3, 5, \cdots . \tag{79}$$

The graph of the odd-harmonic function with this as its Fourier

series is as shown in Fig. 36.   Hence Eq. (79) does not hold for
$x = 0$ or $x = \pi$.   If we replace the $\pi/2$ in Eq. (60) by the
right member of Eq. (64), and note that the terms in parentheses
are the expansions of the summations in Eq. (79), we see that our
three series are consistent for $0 < x < \pi$.

The simplified formulas for the coefficients given in Eqs. (56),
(61), and (75) may be used to find the series of restricted form
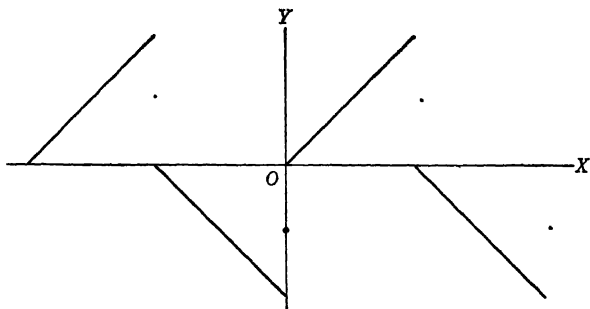of period $p = 2L$ with specified values on the half-range $0,L$, as

Fig. 36.   An odd-harmonic function.

we have shown.   And this is their principal application.   How-
ever, when a function of period $2L$ is to take on specified values
on some full range as $-L,L$ or $0,2L$ we may still use one of these
equations in place of Eq. (48) to (50) if the values are such that
the periodic function is even, odd, or odd-harmonic.   For
example, if our problem was to find a series of period $2L$ represent-
ing $|x|$ in the interval $-L < x < L$, we might notice from its
graph, Fig. 32, that the periodic function is even, and thus com-
pute the coefficients from Eq. (56) as in Eqs. (58) and (59).   Or
we might notice that if the line $y = \pi/2$ were taken as a new x
axis, the graph of Fig. 32 would be odd-harmonic as well as even.
Hence the series for $|x|$ equals the constant $\pi/2$ plus terms
$a_m \cos mx$, whose coefficients could be found from Eq. (75) as in
Eq. (77).   Either method would be simpler than the use of Eq.
(55), since the integrals for $|x|$ from $-L$ to L would have to be
taken for $-x$ from $-L$ to 0, and for x from 0 to L.

To detect from its graph that a function is even, odd, even-harmonic, or odd-harmonic one must recall that the graph for $-L,0$ comes from that for $0,L$ by a displacement for an even-harmonic function, by a displacement together with a reflection in the x axis for an odd-harmonic function, by a reflection about the y axis for an even function, and a reflection about the y axis
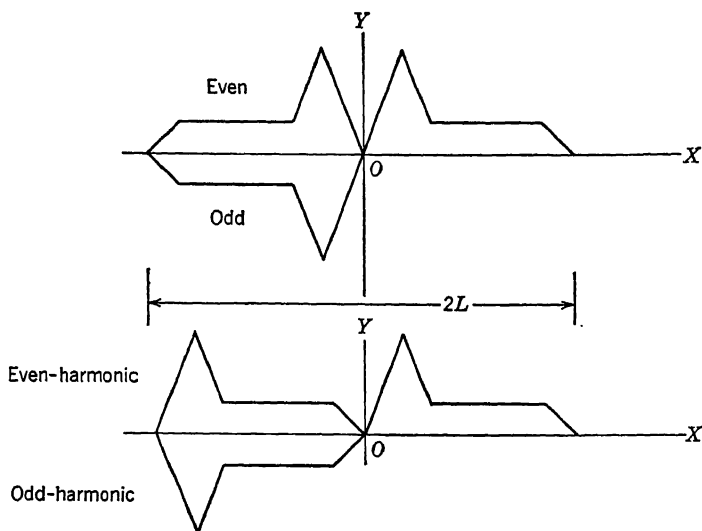
Fig. 37.

together with a reflection about the x axis for an odd function. These geometric relations are summarized in Fig. 37.

EXERCISE IX

Verify that each of the following functions is an even function of period π, and find its Fourier series:

1. $|\sin x|$.   **2.** $|\cos x|$.   **3.** $\sin^2 x$.   **4.** $\cos^2 x$.

Verify that each of the following functions is odd and also odd-harmonic of period 2π, and find its Fourier series:

5. $\sin x |\sin x|$.   **6.** $\sin x |\cos x|$.   **7.** $\sin^3 x$.   **8.** $\sin x \cos^2 x$.

Verify that each of the following functions is even and also odd-harmonic of period 2π, and find its Fourier series:

9. $\cos x \, |\sin x|$.   **10.** $\cos x \, |\cos x|$.   **11.** $\cos x \sin^2 x$.   **12.** $\cos^3 x$.

Find the Fourier cosine series of period 8 which represents $f(x)$ in the interval $0 < x < 4$ when

13. $f(x) = x$ if $0 < x < 4$.   **14.** $f(x) = 2 - x$ if $0 < x < 4$.
15. $f(x) = 0$ if $0 < x < 2$ and $f(x) = 4$ if $2 < x < 4$.
16. $f(x) = x$ if $0 < x < 2$ and $f(x) = 2$ if $2 < x < 4$.

Find the Fourier sine series of period 8 which in the interval $0 < x < 4$ represents the function given in

17. Prob. 13.   **18.** Prob. 14.   **19.** Prob. 15.   **20.** Prob. 16.

Verify that for $0 < x < L$, with $\omega = \pi/L$,

21. $1 = \dfrac{4}{\pi} \left(\sin \omega x + \dfrac{1}{3} \sin 3\omega x + \dfrac{1}{5} \sin 5\omega x + \cdots \right)$.

22. $x = \dfrac{2L}{\pi} \left(\sin \omega x - \dfrac{1}{2} \sin 2\omega x + \dfrac{1}{3} \sin 3\omega x - \cdots \right)$.

23. $x = \dfrac{L}{2} - \dfrac{4L}{\pi^2} \left(\cos \omega x + \dfrac{1}{3^2} \cos 3\omega x + \dfrac{1}{5^2} \cos 5\omega x + \cdots \right)$.

24. From Probs. 21 and 22, deduce that for $0 < x < L$,

$$Ax + B = \frac{1}{\pi} \left[(4B + 2LA) \sin \omega x - \frac{2LA}{2} \sin 2\omega x \right.$$
$$\left. + \frac{4B + 2LA}{3} \sin 3\omega x - \frac{2LA}{4} \sin 4\omega x + \cdots \right].$$

25. Show that the Fourier cosine series of period $2\pi/\omega$ which represents the function $f(x) = 1$ if $0 < x < w/2$ and $f(x) = 0$ if
$$w/2 < x < 2\pi/\omega$$
is
$$\frac{\omega w}{2\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\omega w}{2} \cos n\omega x.$$

26. When an emf $E \sin \omega t$ with $E > 0$ is impressed on a full-wave rectifier, the output voltage $e = E \, |\sin \omega t|$.   Show that

$$e = E \sin \omega t + E \left(\frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2n\omega t}{4n^2 - 1} \right)$$

$$= \frac{2E}{\pi} (1 - 0.67 \cos 2\omega t - 0.13 \cos 4\omega t - 0.06 \cos 6\omega t$$

$$- \cdots).$$

27. Show that the output of a half-wave rectifier, the e of Fig. 29, is $\frac{1}{2}E \sin \omega t + \frac{1}{2}E \, |\sin \omega t|$, and use this and Prob. 26 to check Prob. 28 of Exercise VIII.

28. Check the terms of Prob. 23, other than the constant term, by integrating the terms of Prob. 21.

Use Prob. 24 to show that the following expansions hold for $0 < x < \pi$:

29. $\dfrac{\pi}{4} = \sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dfrac{\sin 7x}{7} + \cdots$.

30. $-\dfrac{x}{2} + \dfrac{\pi}{4} = \dfrac{\sin 2x}{2} + \dfrac{\sin 4x}{4} + \dfrac{\sin 6x}{6} + \cdots$.

Check the terms in x by integration as in Prob. 28, then find the constant term by using Eq. (56) with $L = \pi$, and so derive for $0 < x < \pi$:

31. $-\dfrac{\pi x}{4} + \dfrac{\pi^2}{8} = \cos x + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \dfrac{\cos 7x}{7^2} + \cdots$, by integrating the terms of Prob. 29.

32. $\dfrac{x^2}{4} - \dfrac{\pi x}{4} + \dfrac{\pi^2}{24} = \dfrac{\cos 2x}{2^2} + \dfrac{\cos 4x}{4^2} + \dfrac{\cos 6x}{6^2} + \cdots$, by integrating the terms of Prob. 30.

Show that the following expansions hold for $0 < x < \pi$:

33. $-\dfrac{\pi x^2}{8} + \dfrac{\pi^2 x}{8} = \sin x + \dfrac{\sin 3x}{3^3} + \dfrac{\sin 5x}{5^3} + \dfrac{\sin 7x}{7^3} + \cdots$, by integrating the terms of Prob. 31 from 0 to x.

34. $\dfrac{x^3}{12} - \dfrac{\pi x^2}{8} + \dfrac{\pi^2 x}{24} = \dfrac{\sin 2x}{3^3} + \dfrac{\sin 4x}{4^3} + \dfrac{\sin 6x}{6^3} + \cdots$, by integrating the terms of Prob. 32 from 0 to x.

35. Square both sides of Eq. (47), and take averages for the interval $-L,L$ where $L = \pi/\omega$, and so deduce that

$$\frac{1}{2L} \int_{-L}^{L} [f(x)]^2 dx = a^2 + \frac{1}{2}(a_1{}^2 + b_1{}^2 + a_2{}^2 + b_2{}^2 + a_3{}^2 + b_3{}^2 + \cdots).$$

36. Show that if $f(x)$ is odd, even, odd-harmonic for the period $2L$ or even-harmonic for the period $2L$ we may replace the left member in Prob. 35 by $\frac{1}{L} \int_{0}^{L} [f(x)]^2 dx$, and in the right member omit those terms which equal zero.

37. By using Probs. 29 and 36 deduce the validity of the equation $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$. Check by putting $x = 0$ in Prob. 31.

38. By using Probs. 30 and 36 deduce the validity of the equation $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$. Check by putting $x = 0$ in Prob. 32.

39. By using Probs. 33 and 36 deduce the validity of the equation $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$.

20. Harmonic Analysis

One of the early applications of Fourier expansions concerned the resolution of a musical note into its fundamental and over-tones, or harmonics. For this reason the determination of the Fourier coefficients of a given function is often called *harmonic analysis*. When the function $f(x)$ is complicated, or given by a graph or tabulation, it is not practical to evaluate the integrals in Eqs. (48), (49), and (50) by the usual elementary methods. If available, harmonic analyzers may be used. These are instruments somewhat similar to planimeters by which the coefficients a_n and b_n can be obtained from a plot, to suitable scale, of $f(x)$ itself. There are also approximate numerical methods, one of which we proceed to explain.

Let $y = f(x)$ be of period p. Select some small positive integer r. Divide the interval $0,p$ into $2r + 1$ equal intervals

with end points $0 = x_0,\ x_1,\ x_2,\ \cdots,\ x_{2r} = p$. Find the corresponding values $y_0,\ y_1,\ y_2,\ \cdots,\ y_{2r}$ such that $y_s = f(x_s)$. We shall obtain our harmonic analysis of $f(x)$, or approximation to the Fourier series for $f(x)$, by selecting the coefficients $A,\ A_1,\ A_2,\ \cdots A_r$, and $B_1,\ B_2,\ \cdots,\ B_{r-1}$ so that the equation

$$y = A + \sum_{k=1}^{r} A_k \cos k\omega x + \sum_{k=1}^{r-1} B_k \sin k\omega x, \qquad (80)$$

with $\omega = 2\pi/p$, makes $y = y_s$ when $x = x_s$ for $s = 0,\ 1,\ 2,\ \cdots,\ 2r$.

We have omitted the term $B_r \sin r\omega x$. At the points we are considering, $x_s = sp/2r = s\pi/r\omega$. Hence $r\omega x_s = 2\pi$, so that $\sin r\omega x_s = 0$. Thus our conditions would in no way determine B_r.

The right member of Eq. (80) takes the same value for $x_{2r} = p$ as for $x_0 = 0$, since $p\omega = 2\pi$. But since $f(x)$ is of period p, $y_{2r} = y_0$. Hence if Eq. (80) makes $y = y_0$ when $x = x_0$, it will necessarily make $y = y_{2r}$ when $x = x_{2r}$. And we have imposed essentially $2r$ conditions:

$$y_s = A + \sum_{k=1}^{r} A_k \cos k\omega x_s + \sum_{k=1}^{r-1} B_k \sin k\omega x_s,$$
$$s = 0,\ 1,\ 2,\ \cdots,\ 2r - 1. \qquad (81)$$

The values of the $2r$ unknown coefficients $A,\ A_1,\ \cdots,\ A_r$, $B_1,\ \cdots,\ B_{r-1}$ are determined by these $2r$ simultaneous first-degree equations. They may be found by a procedure similar to that of Sec. 18, using sums in place of integrals. That is, we multiply the $2r$ equations, Eqs. (81), in turn by 1, $\sin q\omega x_s$, $\cos q\omega x_s$ and add the resulting equations for the $2r$ values of s in each case.

For the multiplier 1, on using the summations

$$\sum_{s=0}^{r-1} \cos k\omega x_s = 0, \qquad \text{when } k = 1,\ 2,\ \cdots,\ r, \qquad (82)$$

$$\sum_{s=0}^{r-1} \sin k\omega x_s = 0 \qquad \text{when } k = 1,\ 2,\ \cdots,\ r - 1, \qquad (83)$$

found in Probs. 2 and 3 of Exercise X, we obtain the result

$$\sum_{s=0}^{r-1} y_s = 2rA, \qquad \text{or} \qquad A = \frac{1}{2r}\sum_{s=0}^{r-1} y_s. \tag{84}$$

For the multiplier $\sin q\omega x_s$, on using Eq. (83) and

$$\sum_{s=0}^{r-1} \sin k\omega x_s \sin q\omega x_s = 0 \qquad \text{when } k \neq q,$$
$$= r \qquad \text{when } k = q \text{ and } k \neq r, \tag{85}$$
$$\sum_{s=0}^{r-1} \cos k\omega x_s \sin q\omega x_s = 0, \tag{86}$$

found in Probs. 4 and 6 of Exercise X, we obtain the result

$$\sum_{s=0}^{r-1} y_s \sin q\omega x_s = rB_q, \qquad \text{or} \qquad B_q = \frac{1}{r}\sum_{s=0}^{r-1} y_s \sin q\omega x_s. \tag{87}$$

For the multiplier $\cos q\omega x_s$, on using Eqs. (82), (86), and

$$\sum_{s=0}^{r-1} \cos k\omega x_s \cos q\omega x_s = 0 \qquad \text{when } k \neq q,$$
$$= r \qquad \text{when } k = q \text{ and } k \neq r,$$
$$= 2r \qquad \text{when } k = q = r, \tag{88}$$

found in Prob. 5 of Exercise X, we obtain the results

$$\sum_{s=0}^{r-1} y_s \cos q\omega x_s = rA_q \qquad \text{when } q = 1, 2, \cdots, r-1$$
$$= 2rA_r \qquad \text{when } q = r. \tag{89}$$

When solved for the coefficients, these become

$$A_r = \frac{1}{2r}\sum_{s=0}^{r-1} y_s \cos r\omega x_s, \quad A_q = \frac{1}{r}\sum_{s=0}^{r-1} y_s \cos q\omega x_s \text{ for } q \neq r. \tag{90}$$

Compact schematic arrangements for computing the coefficients from the values of y_s by Eqs. (84), (87), and (90) have been

worked out.   For example, when $r = 6$, we may use the following

SCHEDULE OF HARMONIC ANALYSIS FOR TWELVE ORDINATES

1. Arrange the twelve values y_1, y_2, $\cdots$, $y_{12} = y_0$ as shown, add the two lines for the sums, U_0 to U_6, and subtract the second line from the first for the differences, u_1 to u_5.

	y_1	y_2	y_3	y_4	y_5	y_6
	y_{12}	y_{11}	y_{10}	y_9	y_8	y_7
Sum	U_0	U_1	U_2	U_3	U_4	U_5	U_6
Difference		u_1	u_2	u_3	u_4	u_5

2. Arrange the numbers U_0 to U_6 and u_1 to u_5 as shown, and again add the two lines for the sums, and subtract the second line from the first for the differences.

	U_0	U_1	U_2	U_3		u_1	u_2	u_3
	U_6	U_5	U_4			u_5	u_4	
Sum	V_0	V_1	V_2	V_3		W_1	W_2	W_3
Difference	v_0	v_1	v_2			w_1	w_2	

3. The coefficients are found from the numbers just obtained by the calculations indicated in the following tables:

TABLE FOR SINE COEFFICIENTS

Multiply by						
0.500	W_1					
0.866		W_2	w_1	w_2		
1.000	W_3				W_1	W_3
Add products	I	II	I	II	I	II
I + II	$6B_1$		$6B_2$			
I − II	$6B_5$		$6B_4$		$6B_3$	

TABLE FOR CONSTANT AND COSINE COEFFICIENTS

Multiply by								
0.500			v_2		$-V_2$	V_1		
0.866				v_1				
1.000	$V_0 + V_2$	$V_1 + V_3$	v_0		V_0	$-V_3$	v_0	v_2
Add products	I	II	I	II	I	II	I	II
I + II	$12A$		$6A_1$		$6A_2$			
I − II	$12A_6$		$6A_5$		$6A_4$		$6A_3$	

In these tables, the multipliers are $0.500 = \sin 30°$,

$$0.866 = \sin 60°,$$

$1.000 = \sin 90°$, and because of relations like

$$\sin 30° = \sin 150° = -\sin 210° = -\sin 330°$$
$$= \cos 60° = -\cos 120° = -\cos 240° = \cos 300°,$$

the effect of the schedule is to work out the sums in Eqs. (84), (87), and (90) for $r = 6$.

If the function to be analyzed has a discontinuity at one of the points, x_s, y_s should be taken as $\frac{1}{2}[f(x_s-) + f(x_s+)]$. In particular, when representing a continuous function $f(x)$ with $f(p)$ not equal to $f(0)$ by a Fourier expansion of period p, one should set $y_0 = y_{2r} = \frac{1}{2}[f(0) + f(p)]$.

The computation of the coefficients may be checked by testing Eq. (81) with $r = 6$, $\omega x_s = s30°$, for some of the values $s = 0$ to 11.

EXERCISE X

1. Let $x_s = sp/2r = s\pi/r\omega$ and N be 0 or a positive or negative integer. Show that the sum $\sum_{s=0}^{2r-1} e^{iN\omega x_s} = 2r$ when N is 0, 2r,

$-2r$, or some integral multiple of $2r$ but that the sum is zero when N is not divisible by $2r$.   HINT: Note that all the terms are powers of $e^{2\pi i} = 1$ in the first case, and in the second case recall that the sum of the geometric progression $\displaystyle\sum_{s=0}^{2r-1} R^s = \frac{R^{2r} - 1}{R - 1}$ if $R \neq 1$.

In Probs. 2 through 6, $x_s = sp/2r = s\pi/r\omega$, and all the sums are on s from $s = 0$ to $s = 2r - 1$.   The letters k and q are some one of the integers $0, 1, 2, \cdots, 2r$.   By expressing the sines and cosines in terms of complex exponentials and using Prob. 1, verify that

2. $\Sigma \cos k\omega x_s = 2r$ when $k = 0$ or $k = 2r$. and
$\qquad\qquad = 0$ when $k \neq 0$ or $2r$.
3. $\Sigma \sin k\omega x_s = 0$.
4. $\Sigma \sin k\omega x_s \sin q\omega x_s = r$ when $k = q$, but $k \neq r$,
$\qquad\qquad\qquad = 0$ when $k \neq q$, or when $k = q = r$.
5. $\Sigma \cos k\omega x_s \cos q\omega x_s = r$ when $k = q$, but $k \neq r$
$\qquad\qquad\qquad = 2r$ when $k = q = r$, and
$\qquad\qquad\qquad = 0$ when $k \neq q$.
6. $\Sigma \cos k\omega x_s \sin q\omega x_s = 0$.

7. If $f(x)$ is an odd function of period p, show that the sums $U_0, \cdots, U_6$ and hence the cosine coefficients obtained from the twelve-ordinate scheme of the text will all be zero.   Also show that in this case we may replace the differences u_1 to u_5 by $2y_1$ to $2y_5$.

Use Prob. 7. to find approximate values of A_1 to A_5 in each of the following cases.   Each function is assumed to be odd and of period $p = 2\pi/\omega$, and the values are given for $x_s = ps/12$.

x	$p/12$	$2p/12$	$3p/12$	$4p/12$	$5p/12$
8. y	0.1	0.2	0.3	0.4	0.5
9. y	1.0	1.0	1.0	1.0	1.0
10. y	0.2	0.4	0.6	0.4	0.2
11. y	2.00	6.93	10.00	6.93	2.00

12. If an emf $e = E_0 \sin \omega t$ is impressed on a tube whose characteristic output can be represented by $i = ae + be^3$, show that the output current $i = E_0(a + \frac{3}{4}b) \sin \omega t - \frac{1}{4}E_0 b \sin 3\omega t$.

13. Use Prob. 12, with $E_0 = 1, a = 2, b = 8$ to check Prob. 11.

14. When an emf $e = E_0 \sin \omega t$ is impressed on a certain tube, its characteristic output current i is an odd and odd-harmonic function. If for $t = 0, 30°, 60°, 90°$ the values of i are $0, a, b, c$, respectively, show that the harmonic analysis of i is

$$i = \frac{a + c + b\sqrt{3}}{3} \sin \omega t + \frac{2a - c}{3} \sin 3\omega t$$
$$+ \frac{a + c - b\sqrt{3}}{3} \sin 5\omega t.$$

15. Use Prob. 14 to check Probs. 9, 10, and 11.

21. Complex Fourier Series

It is often convenient to write the Fourier series (47) in complex form. We note from Sec. 2 that

$$\cos k\omega x = \frac{1}{2}(e^{i\omega x} + e^{-i\omega x}), \qquad \sin k\omega x = \frac{1}{2i}(e^{i\omega x} - e^{-i\omega x}). \quad (91)$$

It follows from these equations that

$$a_k \cos k\omega x + b_k \sin k\omega x = C_k e^{ik\omega x} + C_{-k} e^{-ik\omega x}, \qquad (92)$$

provided that we define

$$C_k = \frac{a_k - ib_k}{2}, \qquad C_{-k} = \frac{a_k + ib_k}{2}. \qquad (93)$$

If we further define $C_0 = a$, by combining Eqs. (92) and (47), we obtain the compact expression for the Fourier series for $f(x)$

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega x}. \qquad (94)$$

This represents $f(x)$ in the interval $c, c + p$ in such a way that correct relations may be obtained from this equation by termwise

integration after multiplication by any function of x. If we use $e^{in\omega x}$ as the multiplier, where n is 0 or any positive or negative integer, integrate from c to $c + p$, and use the result of Prob. 2 of Exercise XI, we find that

$$C_n = \frac{1}{p} \int_c^{c+p} f(x)e^{-in\omega x} \, dx. \tag{95}$$

The right member of Eq. (94) is called the *complex Fourier series* for the real function $f(x)$, and Eq. (95) is the *complex Fourier coefficient formula*. Since

$$e^{-in\omega x} = \cos n\omega x - i \sin n\omega x, \tag{96}$$

for real $f(x)$ the real and imaginary parts of Eq. (95) reduce to Eqs. (48), (49), and (50). Here C_n and C_{-n} are conjugate complex numbers.

If we applied Eqs. (94) and (95) to a complex function of the real variable $x, f = f_1 + if_2$, our expansion would be equivalent to the ordinary Fourier series for f_1 plus i times that for f_2.

Let $g(x)$ be a second real function of period p, whose complex Fourier series is

$$g(x) = \sum_{k=-\infty}^{\infty} D_k e^{ik\omega x}. \tag{97}$$

Then by using the result of Prob. 2 of Exercise XI we find for the average value of the product $f(x)g(x)$ over the interval $0,p$

$$\frac{1}{p} \int_0^p f(x)g(x)dx = \sum_{k=-\infty}^{\infty} C_k D_{-k}. \tag{98}$$

And in particular

$$\frac{1}{p} \int_0^p [f(x)]^2 \, dx = \sum_{k=-\infty}^{\infty} C_k C_{-k}. \tag{99}$$

These results can be used in finding the average power or the rms values when the currents and emfs are expressed as complex Fourier series.

22. The Fourier Integral

Let $f(x)$ be a real function defined for all values of x and piecewise regular on any finite interval. Instead of requiring $f(x)$ to be periodic, we are here considering functions which are small for numerically large values of x, so that they approach zero when x tends to plus infinity or to minus infinity. As in Sec. 19, for any value of L we may find a Fourier series of period $2L$ which represents $f(x)$ on the interval $-L$ to L. Its complex form, as in Eq. (94), is

$$f(x) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega x}, \qquad -L < x < L, \qquad (100)$$

where

$$C_k = \frac{1}{2L} \int_{-L}^{L} f(t) e^{-ik\omega t}\, dt, \qquad \omega = \frac{\pi}{L} \qquad (101)$$

which is Eq. (95) with p, c, n, x replaced by $2L$, $-L$, k, t, respectively. We may think of the infinite series in Eq. (100) as the limit of the finite sum from $-n$ to n when n tends to infinity. This leads us to write

$$f(x) = \lim_{n \to \infty} S_n \qquad (102)$$

and to study the partial sum S_n given by

$$S_n = \sum_{k=-n}^{n} C_k e^{ik\omega x} = \sum_{k=-n}^{n} \left[\frac{1}{2L} \int_{-L}^{L} f(t) e^{-ik\omega t}\, dt \right] e^{ik\omega x}. \qquad (103)$$

When we interchange the order of finite summation and integration, and bring the exponential in x inside the integral, this becomes

$$S_n = \frac{1}{2L} \int_{-L}^{L} \left[\sum_{k=-n}^{n} e^{ik\omega(x-t)} \right] f(t)\, dt \qquad (104)$$

We now introduce the notation

$$k\omega = u_k, \qquad \Delta u_k = \omega = \frac{\pi}{L} \qquad (105)$$

Then the expression for the partial sum S_n may be written

$$\frac{1}{2\pi} \sum_{k=-n}^{n} \Delta u_k \int_{-L}^{L} e^{iu_k(x-t)} f(t) dt. \tag{106}$$

We next put $n = AL/\pi$, and let L become infinite. Then this makes $\Delta u_k \to 0$, and the expression (106) suggests the integral

$$\frac{1}{2\pi} \int_{-A}^{A} du \int_{-\infty}^{\infty} e^{iu(x-t)} f(t) dt. \tag{107}$$

In fact, if we let n and L increase in such a way that $A = n\pi/L$ remained constant, the expression (106) would approach the repeated integral (107) as a limit.

By Eq. (102) the limit of the expression (106) as $n \to \infty$ with L fixed is $f(x)$ in the interval $-L$ to L. Since this makes $n\pi/L$ tend to infinity, we are led to take the limit of the expression (107) as A tends to infinity and to conjecture that under suitable conditions

$$f(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} du \int_{-\infty}^{\infty} e^{iu(x-t)} f(t) dt. \tag{108}$$

As a matter of fact, if

$$\int_{-\infty}^{\infty} |f(x)| \, dx \text{ is finite,} \tag{109}$$

$f(x)$ is piecewise regular, and is so defined at points of discontinuity that

$$f(x) = \frac{f(x+) + f(x-)}{2}, \tag{110}$$

then it may be proved that Eq. (108) holds for all values of x. This is the complex form of the *Fourier integral theorem*.

Our discussion made Eq. (108) plausible if the extension from $-A$ to A gave the same result as a slightly different limiting process, and the proof which we omit consists in showing that the contributions from $-\infty$ to $-A$ and from A to ∞ have no effect on the limiting situation when Eq. (109) holds. As we have stated it, the theorem holds if $f = f_1 + if_2$ is a complex function of the real variable x.

When $f(x)$ is a real function, the imaginary part of the right member of Eq. (108) is zero, since we take the integral from $-A$ to A of a function of u which is odd because of the presence of the factor $\sin u(x - t)$.   Taking real parts leads to

$$f(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} du \int_{-\infty}^{\infty} \cos u(x - t) f(t) dt$$
$$= \frac{1}{\pi} \int_{0}^{\infty} du \int_{-\infty}^{\infty} \cos u(x - t) f(t) dt, \qquad (111)$$

since the presence of $\cos u(x - t)$ makes the first integral an even function of u.   This is the real form of the *Fourier integral theorem.*

23. Fourier Transforms

For a function satisfying the conditions imposed on $f(x)$ in the preceding section, we define the *Fourier transform* by the relation

$$F(u) = \int_{-\infty}^{\infty} e^{-iut} f(t) dt. \qquad (112)$$

The function $f(x)$ may be recovered from its transform since by Eq. (108)

$$f(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} e^{iux} F(u) du. \qquad (113)$$

This relation expresses $f(x)$ in terms of functions e^{iux} of frequency u by an integration on u, over a continuous range of values.   It is somewhat analogous to the expression of a periodic function in terms of functions $e^{ik\omega x}$ by a sum on k, Eq. (94), and thus for a discrete set of frequencies k.

For many linear systems, the response $R(x,u)$ to a single complex exponential e^{iux} is known or can be found easily.   For such systems, the response to the function $f(x)$ of Eqs. (112) and (113) may be found by building up a function $R(x)$ from $R(x,u)$ by the same process that $f(x)$ comes from e^{iux} in Eq. (113).   Thus

$$R(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} R(x,u) F(u) du. \qquad (114)$$

The direct exact evaluation of the infinite integrals involved in this process is usually not practicable, but in some cases their evaluation may be reduced to cases already tabulated. In a number of applications, it is more convenient to use the modification to Laplace transforms instead of the Fourier transforms. The Laplace transforms are discussed in the following section.

24. Laplace Transforms

If a constant voltage is impressed on a circuit, starting at $t = 0$, we may think of the impressed emf as the function of the time

$$e(t) = 0 \quad \text{for } t = 0 \quad \text{and} \quad e(t) = E_0 \quad \text{for } t > 0. \quad (115)$$

or, for a suddenly impressed alternating emf

$$e(t) = 0 \quad \text{for } t = 0 \quad \text{and} \quad e(t) = E_0 \sin \omega t \quad \text{for } t > 0 \quad (116)$$

These functions, which are zero for negative values and whose numerical value is at most a constant, E_0, are special cases of functions $g(x)$ satisfying the following conditions:

$$g(x) = 0 \quad \text{for } x < 0, \quad |g(x)| < e^{ax} \quad \text{for } x > x_1, \quad a > 0, \quad (117)$$

and $g(x)$ is piecewise regular.

These functions need not satisfy the condition (109) and thus the definition of a Fourier transform (112) may not be directly applicable to them. For example, the function $g(x)$ corresponding to Eq. (115) with $E_0 = 1$ is

$$g(x) = 0 \quad \text{for } x < 0 \quad \text{and} \quad g(x) = 1 \quad \text{for } x > 0, \quad (118)$$

and for this function

$$\int_{-\infty}^{\infty} g(x)dx = \lim_{M \to \infty} \int_0^M 1dx = \lim_{M \to \infty} M = \infty, \quad (119)$$

so that Eq. (109) does not hold. Also

$$\int_{-\infty}^{\infty} e^{-iut}g(t)dt = \int_0^{\infty} e^{-iut}1dt = \frac{e^{-iut}}{-iu}\Big|_0^{\infty}$$

$$= \lim_{M \to \infty} \frac{1}{u}[\sin Mu + i(\cos Mu - 1] \quad (120)$$

which oscillates and so diverges. However, for the modified function

$$f(x) = g(x)e^{-bx}, \qquad b > a, \tag{121}$$

condition (109) will necessarily hold, and we may form

$$F(u) = \int_{-\infty}^{\infty} e^{-iut}f(t)dt = \int_0^{\infty} e^{-(b+iu)t}g(t)dt. \tag{122}$$

Let us put $p = b + iu$, and denote the resulting function of p as derived from $g(x)$ by $G(p)$. Then

$$G(p) = \int_0^{\infty} e^{-pt}g(t)dt. \tag{123}$$

The function $G(p)$ so defined is called the *Laplace transform* of $g(x)$. The variable p is complex, but when its real part b is greater than some possible a for which Eq. (117) holds, $G(p)$ is an analytic function of p. Hence we may often find the form of $G(p)$ by assuming that p is real and sufficiently large for the integral in Eq. (123) to converge and calculating that integral.

Some examples and properties of Laplace transforms are given in Exercise XI. Additional examples and their application will be found in Chap. 5. In these applications we solve problems by operations which eventually determine the Laplace transform of the solution. Hence it is necessary to know to what extent $G(p)$ determines $g(x)$. We assume that $g(x)$ is piecewise regular and satisfies the condition (117). The solutions to the problems of the type we consider always have these properties. Thus the $f(x)$ of Eq. (121) is a function for which the Fourier integral theorem or Eqs. (112) and (113) hold. And for the $F(u)$ of Eq. (122) we have

$$g(x)e^{-bx} = f(x) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} e^{iux}F(u)du. \tag{124}$$

Put $p = b + iu$. Then $F(u)$ equals the $G(p)$ of Eq. (123). Also $dp = i\,du$, and $p = b + iA$ when $u = A$, $p = b - iA$ when $u = \cdot\cdot A$. If we multiply by e^{bx} and use these relations, Eq. (124) becomes

$$g(x) = \lim_{A \to \infty} \frac{1}{2\pi i} \int_{b-iA}^{b+iA} e^{px}G(p)dp. \tag{125}$$

This relation necessarily holds for x corresponding to an interior point of one of the regular arcs. If there are points of discontinuity, the relation would hold at these points if

$$g(x) = \frac{g(x+) + g(x-)}{2}. \tag{126}$$

It follows that, if the Laplace transform $G(p)$ is given, the function $g(x)$, required to satisfy Eq. (126), is uniquely determined at all points.

In practice, the actual values of $g(x)$ at the points of discontinuity are not important. Sometimes we take them as $g(x+)$ or $g(x-)$. More often we leave them undefined. They have no effect on the value of $G(p)$ as determined by Eq. (123). And the values of $g(x)$ on the regular arcs are uniquely determined by $G(p)$.

The right member of Eq. (125) is the same for all sufficiently large values of b. In some cases it can be found by the method of residues from the theory of functions of a complex variable. However, in our problems we will determine the correspondence of $G(p)$ to $g(x)$ by Eq. (123) combined with a few general principles for a few simple functions, tabulate the results, and find $g(x)$ from $G(p)$ by reading the table backward. Thus we need Eq. (125) only to justify this process by showing that there is just one $g(x)$ for a given $G(p)$.

25. References

Alternative elementary discussions of Fourier series will be found in Churchill's *Fourier Series and Boundary Value Problems*, and in Chaps. II and VIII of the author's *Differential Equations for Electrical Engineers*.

For an extended discussion of harmonic analysis, the reader may consult Lipka's *Graphical and Mechanical Computation*, Vol. II, or Scarborough's *Numerical Analysis*.

Derivations of most of the results which we have stated without proof will be found in Carslaw's *Theory of Fourier Series*

and Integrals, or in Chap. XIV of the author's *Treatise on Advanced Calculus.*

EXERCISE XI

1. By the procedure used to derive Eq. (25), show that the integral $\int_c^{c+p} e^{ik\omega x}\, dx = p$ if $k = 0$, and $= 0$ if k is an integer, positive or negative, where $p = 2\pi/\omega$.

2. With the notation of Prob. 1, deduce from Prob. 1 that the integral $\int_c^{c+p} e^{i(k-n)\omega x}\, dx = p$ if $n = k$, and $= 0$ if n is 0 or a positive or negative integer with $n \neq k$.

3. By expressing the sines in terms of complex exponentials, deduce the result of Prob. 29 of Exercise VII from Eqs. (98) and (99).

4. Let $f(x) = 0$ for $x < -a$, $f(x) = 1$ for $-a < x < a$, $f(x) = 0$ for $x > a$. Show that for this function the Fourier transform $F(u)$ of Eq. (112) is $F(u) = \dfrac{2}{u} \sin au$.

5. If $f(x)$ is a real even function of x, show that the Fourier transform of Eq. (112) is the real even function of u, given by the expression $F(u) = \int_{-\infty}^{\infty} \cos ut\, f(t)dt = 2\int_0^{\infty} \cos ut\, f(t)dt$.

6. Use Prob. 5 to check Prob. 4.

7. When the conditions of Prob. 5 hold, so that $F(u)$ is real and even, show that Eq. (113) is equivalent to

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos ux\, F(u)du.$$

8. By applying the result of Prob. 7 to the function of Prob. 4, with $x = 0$, show that $\int_0^{\infty} \dfrac{\sin au}{u}\, du = \dfrac{\pi}{2}$, if $a > 0$.

9. Show that $\int_0^{\infty} \dfrac{\sin au}{u}\, du = \dfrac{\pi}{2}$ for $a > 0$, $= 0$ for $a = 0$, and $= -\dfrac{\pi}{2}$ for $a < 0$. HINT: Note that the integrand is zero when $a = 0$ and reverses sign when a reverses sign, and use Prob. 8.

10. Let $f(x) = 0$ for $x < a$, $f(x) = 1$ for $a < x < b$, $f(x) = 0$ for $x > a$. Show that for this function the Fourier transform $F(u)$ of Eq. (112) is $F(u) = (e^{-iua} - e^{-iub})/iu$. Also show that Eq. (113) is equivalent to

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[\frac{\sin u(x - a)}{u} - \frac{\sin u(x - b)}{u} \right] du.$$

11. Use Prob. 9 to verify directly that in the result of Prob. 10 the right member equals $\frac{1}{2}$ for $x = a$ or $x = b$ and for other values of x equals the values given in the definition of $f(x)$.

12. Let c_1 and c_2 be constants, $G_1(p)$ be the Laplace transform of $g_1(x)$ and $G_2(p)$ be the Laplace transform of $g_2(x)$. Show that the Laplace transform of $c_1g_1(x) + c_2g_2(x)$ is $c_1G_1(p) + c_2G_2(p)$.

13. If $g(x) = 0$ for $x < 0$ and $g(x) = 1$ for $x > 0$, show that the Laplace transform $G(p) = 1/p$.

14. If $g(x) = 0$ for $x < 0$ and $g(x) = e^{kx}$ for $x > 0$, show that the Laplace transform $G(p) = 1/(p - k)$.

15. The *convolution* or *faltung* of two functions $f_1(x)$ and $f_2(x)$ is defined by $h(x) = \int_{-\infty}^\infty f_1(y)f_2(x - y)dy$. By putting $y = x - z$, show that $h(x) = \int_{-\infty}^\infty f_2(z)f_1(x - z)dz$ so that the order of the functions is unimportant.

16. If $g_1(x)$ and $g_2(x)$ are each $= 0$ for $x < 0$, show that their convolution as defined in Prob. 15 is $h(x) = \int_0^x g_1(y)g_2(x - y)dy$.

17. The Laplace transform $H(p)$ of the function $h(x)$ of Prob. 16 is $H(p) = \int_0^\infty e^{-pt} dt \int_0^t g_1(y)g_2(t - y)dy$. Since $g_1(y) = 0$ when $y < 0$, and $g_2(t - y) = 0$ when $y > t$ or when $t < 0$ and $y > 0$, this is equal to

$$\int_{-\infty}^\infty e^{-pt} dt \int_{-\infty}^\infty g_1(y)g_2(t - y)dy$$
$$= \int_{-\infty}^\infty g_1(y)dy \int_{-\infty}^\infty e^{-pt} g_2(t - y)dt.$$

By putting $z = t - y$, $t = z + y$ in the integral in t, deduce that $H(p) = \int_{-\infty}^\infty e^{-py}g_1(y)dy \int_{-\infty}^\infty e^{-pz}g_2(z)dz = G_1(p)G_2(p)$. When the functions $g_1(x)$ and $g_2(x)$ each satisfy the conditions (117), all the integrals converge and the change of order of integration is

legitimate, so that the Laplace transform of the convolution is the product of the separate Laplace transforms.

18. Let $g_1(x) = 0$ for $x < 0$ and $g_1(x) = e^x$ for $x > 0$. Also let $g_2(x) = 0$ for $x < 0$ and $g_2(x) = e^{2x}$ for $x > 0$. Show that the convolution $h(x)$ of Probs. 15 and 16 is $e^{2x} - e^x$.

19. By Prob. 14, the Laplace transforms of the functions in Prob. 18 are $G_1(p) = 1/(p - 1)$ and $G_2(p) = 1/(p - 2)$. The Laplace transforms of their convolution $h(x)$, or $H(p)$ is $G_1(p)G_2(p)$ by Prob. 17, and is $G_2(p) - G_1(p)$ by Probs. 12 and 18. Show that these are equal.

CHAPTER 3

PARTIAL DIFFERENTIAL EQUATIONS

In many situations it is necessary to study physical quantities which depend on more than one independent variable. Consideration of the effect of changing these independent variables one at a time leads to the notion of partial derivatives. And known physical principles may be formulated in equations that involve partial derivatives of the unknown functions. Such equations are called partial differential equations.

As a typical example of how these equations arise in engineering or physical problems, we shall discuss in detail the derivation of the equation governing the flow of heat. We shall also describe the equations which dominate a number of other fields, explaining the significance of the quantities which occur in them and giving some indication of the physical laws of which they are the mathematical interpretation.

For some partial differential equations it is possible to find a general solution containing arbitrary functions. We illustrate this, and show how it is also possible to form partial differential equations from an assumed general solution of special type. Finally we explain how to find a form of particular solution which will prove useful in solving the boundary value problems of Chap. 4.

26. Heat Flow

Let us study the flow of heat in a long, thin, uniform rod whose sides are insulated. At any cross section C, Fig. 38, the temperature $U°$C. will depend only on the distance $OC = x$ cm. and the time t sec. Consider a small segment of the rod CC', of length Δx. Then the rate at which heat flows into the segment through the cross section at C is $-KA \dfrac{\partial U}{\partial x}$ cal./sec. The constant K

100

is the thermal conductivity in calories per second per degree centigrade. A is the area of the cross section in square centimeters. And $\dfrac{\partial U}{\partial x}$ is evaluated at x to give the rate of increase of temperature in a direction perpendicular to the cross section, and into the segment. The minus sign is due to the fact that heat flows from a higher to a lower temperature. Similarly the rate of flow of heat out of the segment through the cross section at C' is $-KA \left. \dfrac{\partial U}{\partial x} \right|_{x+\Delta x}$, where the subscript indicates that $\dfrac{\partial U}{\partial x}$ is evalu-
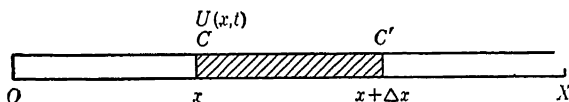
FIG. 38.

ated with x replaced by $x + \Delta x$. The income rate minus the outgo rate is

$$KA \left(\left. \frac{\partial U}{\partial x} \right|_{x+\Delta x} - \left. \frac{\partial U}{\partial x} \right|_{x} \right). \qquad (1)$$

This is the rate at which heat is absorbed by the segment.

Let c cal./gm. °C. be the specific heat and D gm./cm.³ be the density. Then the rate at which heat is absorbed by the segment of volume $A \Delta x$ cm.³, owing to changing temperature, is $cDA \Delta x \left. \dfrac{\partial U}{\partial t} \right|_{x'}$ cal./sec. The rate $\dfrac{\partial U}{\partial t}$ should be an average for the whole segment, and this average will be reached at some point x' between x and $x + \Delta x$. If we equate the expression just found to that obtained in Eq. (1), and divide both sides by $KA \Delta x$, the result is

$$\frac{cD}{K} \left. \frac{\partial U}{\partial t} \right|_{x'} = \frac{\left. \dfrac{\partial U}{\partial x} \right|_{x+\Delta x} - \left. \dfrac{\partial U}{\partial x} \right|_{x}}{\Delta x}. \qquad (2)$$

Now let Δx approach zero. Then x', between x and $x + \Delta x$, approaches x. And in the right member we may put $\dfrac{\partial U}{\partial x} = F(x,t)$

to obtain

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x, t) - F(x,t)}{\Delta x} = \frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial x}\right) = \frac{\partial^2 U}{\partial x^2}, \quad (3)$$

by the definition of the partial derivative of $F(x,t)$ with respect to x. Hence the limiting form of Eq. (2) is

$$\frac{cD}{K}\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}. \quad (4)$$

We need no subscripts since all the derivatives are evaluated at x. If we replace $K/(cD)$ by a^2 cm.²/sec., the equation for heat flow in the rod may be written

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}. \quad (5)$$

If the rod were of homogeneous material, but of variable cross section, we would have to keep A with $\frac{\partial U}{\partial x}$ to be evaluated at x and $x + \Delta x$ in (1). And in this case the final equation would be

$$\frac{\partial U}{\partial t} = a^2 \frac{1}{A}\frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right), \quad (6)$$

for a rod of variable cross section $A(x)$.

For flow in three dimensions, a similar analysis for a rectangular element with dimensions Δx, Δy, Δz would lead to an income minus outgo term for each direction. And the final equation analogous to Eq. (5) would be

$$\frac{\partial U}{\partial t} = a^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right). \quad (7)$$

This same equation holds for diffusion, whether of a solid through a solution, a liquid through a porous material, or one fluid through another. In fact, if U is the concentration of the diffused substance, and a^2 cm.²/sec. is the diffusivity, Eq. (7) is the equation governing diffusion phenomena.

For steady flow of heat the temperature U does not vary with

the time. Hence $\dfrac{\partial U}{\partial t} = 0$ in Eq. (7) and

$$\frac{\partial^2 U}{\partial^2 x} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0, \tag{8}$$

Laplace's equation, is satisfied by the steady-state temperature.

Next consider the steady flow of heat in a rod of variable cross section $A(x)$. Then $\dfrac{\partial U}{\partial t} = 0$ in Eq. (6), and since U is now a function of a single variable x, the partial derivatives reduce to ordinary derivatives, and we may write $\dfrac{d}{dx}$ in place of $\dfrac{\partial}{\partial x}$. Consequently we have

$$0 = a^2 \frac{1}{A} \frac{d}{dx}\left(A \frac{dU}{dx} \right) \quad \text{or} \quad \frac{d}{dx}\left(A \frac{dU}{dx} \right) = 0. \tag{9}$$

This admits the integral

$$A \frac{dU}{dx} = c_1, \quad \text{or} \quad dU = c_1 A \, dx. \tag{10}$$

The physical meaning of the first relation is that the rate at which heat flows through any cross section,

$$Q = -KA \frac{dU}{dx} = -Kc_1, \tag{11}$$

is constant for steady flow. The second relation (10) may be integrated to give

$$U - U_1 = c_1 \int_{x_1}^{x} A \, dx. \tag{12}$$

If the temperature difference for two points on the rod is known, c_1 may be found from Eq. (12) and then the rate of flow through any section Q may be found from Eq. (11).

EXERCISE XII

1. A rod of uniform cross section has temperature U_1 at a point $x = x_1$ and a temperature U_2 at a point $x = x_2$. Show that the

rate of flow through any cross section of area A is

$$Q = -KA\, \frac{U_2 - U_1}{x_2 - x_1}.$$

2. Compute the heat loss per day through 50 m². of brick wall ($K = 0.0020$), if the wall is 30 cm. thick, the inner face is at 20°C., and the outer face is at 0°C. If the combustion of coal is 7000 cal./gm., and the efficiency of the furnace is 60 per cent, how much coal must be consumed daily to compensate for this loss?

3. A refrigerator with walls 8 cm. thick has as its outside dimensions 108 by 108 by 58 cm. The temperature inside is 10°C., and that outside is 25°C., while $K = 0.0002$ is an average value for the walls. Assume that there is uniform flow, and neglect the effects at the edges and corners. That is, regard the heat gain as that for a single wall equal in area to the surface half-way between the inner and outer surfaces or the surface of a box 100 by 100 by 50 cm. Find the heat gain per day.

4. If ice is used in the box of Prob. 3, find the number of kilo-grams required per day, recalling that a gram of ice, in melting, absorbs 80 cal. and the specific heat of water is 1, and assuming that the water from the ice is at 5°C. when it leaves the box.

5. Suppose that a mechanical refrigerating unit is used in the box of Prob. 3. Assume that it pumps 50 per cent as much heat outside the refrigerator as the same electrical energy would generate in a heating coil for which case each watt would produce 0.24 cal./sec. Find the number of kilowatt hours used per day.

6. Carry out the analysis of the text for flow in a rod for which c, D, K, and A, though constant for each cross section, vary with x, and derive the equation for this case

$$c(x)\, D(x)\, A(x)\, \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}\left[K(x)\, A(x)\, \frac{\partial U}{\partial x} \right].$$

7. For the flow of heat along the radii normal to a set of concentric cylinders, deduce the equation:

$$\frac{\partial U}{\partial t} = a^2\, \frac{1}{r}\, \frac{\partial}{\partial r}\left(r\, \frac{\partial U}{\partial r} \right).$$

8. If the flow of Prob. 7 is steady, show that the rate of flow through a cylindrical surface of radius r and length L is

$$Q = -K2\pi rL \frac{dU}{dr}.$$

9. If in Prob. 8 the temperature is U_1 for $r = r_1$ and U_2 for $r = r_2$, show that the rate of flow in calories per second through any cylindrical section of length L is given by

$$Q = -2\pi KL \frac{U_2 - U_1}{\ln r_2 - \ln r_1}.$$

10. A steam pipe 30 cm. in diameter is insulated by a layer of concrete ($K = 0.0025$) 10 cm. thick. The outer surface is at 25°C., and the inner surface is at 155°C. Compute the heat loss in calories per day for 40 m. of pipe. HINT: Use Prob. 8.

11. For the flow of heat along the radii normal to a set of concentric spheres, deduce the equation

$$\frac{\partial U}{\partial t} = a^2 \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right).$$

12. If the flow in Prob. 11 is steady show that the rate of flow through a spherical surface of radius r is

$$Q = -4\pi Kr^2 \frac{dU}{dr}.$$

13. If in Prob. 12 the temperature is U_1 for $r = r_1$ and U_2 for $r = r_2$, show that the rate of flow in calories per second through any spherical section of radius r is given by

$$Q = 4\pi Kr_1 r_2 \frac{U_2 - U_1}{r_2 - r_1}.$$

14. A hollow lead sphere whose inner and outer diameters are 2 cm. and 8 cm. is heated by a small resistance coil placed inside. At what rate must heat be supplied to keep the inner surface at a temperature 15°C. higher than that of the outside surface if $K = 0.0827$ for lead?

15. A spherical shell for which $K = 0.0025$ of inner radius 24 cm. and outer radius 26 cm. has its inner surface 40°C. higher than its outer one. Compute the rate in calories per second at which heat must be supplied by using Prob. 12.

16. Check Prob. 15 by considering the loss as equal to that for a single wall equal in area to the midsurface of the shell, a sphere of radius 25 cm.

17. A long cylindrical shell of material for which $K = 0.003$ of inner radius 100 cm. and outer radius 102 cm. has its inner surface 50°C. higher than its outer one. Compute the rate of heat loss in calories per second, per meter length of the shell, by using Prob. 8.

18. Check Prob. 17 by considering the loss as equal to that for a single wall equal in area to the midsurface of the shell, or lateral surface of a cylinder 101 cm. in radius and 100 cm. in height.

27. Direct Integration

The solution of certain partial differential equations may be obtained by integrating with respect to a single variable, keeping the outer independent variable fixed. For example, consider

$$\frac{\partial z}{\partial x} = 4y, \tag{13}$$

where z is a function of x and y. Keeping y fixed, and integrating with respect to x leads to

$$z = 4xy + f(y). \tag{14}$$

Since y is kept fixed, the "constant" of integration may involve y and therefore is written as an arbitrary function of y, or $f(y)$.

Similarly to solve the equation

$$\frac{\partial^2 z}{\partial y^2} = 4y, \tag{15}$$

we would integrate twice with respect to y, keeping x fixed. The

results would be

$$\frac{\partial z}{\partial y} = 2y^2 + f(x) \qquad \text{and} \qquad z = \frac{2}{3} y^3 + yf(x) + g(x). \quad (16)$$

For the equation

$$\frac{\partial^2 z}{\partial x \, \partial y} = 4y, \qquad \text{or} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 4y, \quad (17)$$

an integration with respect to x gives

$$\frac{\partial z}{\partial y} = 4xy + F(y), \quad (18)$$

and a second integration with respect to y gives

$$z = 2xy^2 + \int^y F(y)dy + g(x) \text{ or } z = 2xy^2 + f(y) + g(x). \quad (19)$$

We may use the simplified notation $f(y)$, since if $F(y)$ is an arbitrary function of y, so is its integral. The omission of a lower limit means that any indefinite integral is to be used.

In a few cases, the solutions of partial differential equations may be found from their similarity to ordinary differential equations of solvable type. Thus consider

$$\frac{\partial^2 z}{\partial x^2} = -y^2 z. \quad (20)$$

If y were a constant k, this would be

$$\frac{d^2 z}{dx^2} = -k^2 z, \qquad \text{with solution } z = c_1 \sin kx + c_2 \cos kx. \quad (21)$$

Replacing c_1 and c_2 by functions of y, and with $k = y$, we find

$$z = f(y) \sin xy + g(y) \cos xy \quad (22)$$

as the solution of Eq. (20).

Suppose we have a system of simultaneous partial differential equations. If any one of them can be solved by the methods just discussed, the restrictions on the arbitrary functions in the solution may be found by substituting the solution in the other equa-

tions of the system. If inconsistencies arise, the system has no solution.

As an example, suppose that we have the equations

$$\frac{\partial^2 z}{\partial x^2} = 0, \qquad \frac{\partial^2 z}{\partial y^2} = 0, \tag{23}$$

which hold simultaneously. By integrating the first equation with respect to x twice, we find that its solution is

$$z = xf(y) + g(y). \tag{24}$$

We next substitute this solution in the second equation, obtaining

$$xf''(y) + g''(y) = 0. \tag{25}$$

For this to be true for all values of x, we must have

$$f''(y) = \frac{d^2 f}{dy^2} = 0 \qquad \text{and} \qquad g''(y) = \frac{d^2 g}{dy^2} = 0. \tag{26}$$

The solutions of these ordinary differential equations are

$$f(y) = c_1 y + c_2, \qquad g(y) = c_3 y + c_4, \tag{27}$$

respectively. From Eqs. (27) and (24) it follows that

$$z = c_1 xy + c_2 x + c_3 y + c_4. \tag{28}$$

This is the solution of the system (23). Thus the solution involves arbitrary constants, but no arbitrary functions. If the second equation had been $\frac{\partial^2 z}{dy^2} = 2x^3$, the system would not have had any solution.

EXERCISE XIII

Integrate each of the following partial differential equations:

1. $\dfrac{\partial z}{\partial x} = 0.$ 2. $\dfrac{\partial z}{\partial x} = 4xy.$ 3. $\dfrac{\partial z}{\partial y} = 0.$

4. $\dfrac{\partial z}{\partial y} = 4y.$ 5. $\dfrac{\partial z}{\partial x} = 3x^2 + 3y^2.$ 6. $\dfrac{\partial z}{\partial y} = \cos \dfrac{y}{x}.$

Solve each of the following partial differential equations:

7. $\dfrac{\partial^2 z}{\partial x\,\partial y} = 0.$ **8.** $\dfrac{\partial^2 z}{\partial x\,\partial y} = 2x + 4y.$

9. $\dfrac{\partial^2 z}{\partial x\,\partial y} = e^{2x-v}.$ **10.** $\dfrac{\partial^2 u}{\partial x\,\partial t} = \dfrac{2x}{t}.$

11. $\dfrac{\partial^2 u}{\partial t^2} = 12x^2 t.$ **12.** $\dfrac{\partial^2 u}{\partial x^2} = 16te^{2x}.$

13. $\dfrac{\partial^2 z}{\partial x^2} = \sin(2x - 3y).$ **14.** $\dfrac{\partial^2 z}{\partial y^2} = 12xy.$

15. $\dfrac{\partial^2 u}{\partial x\,\partial p} = 12px.$

For each of the following equations, introduce a new variable as indicated. Verify that this reduces the equation to the second form, and by integrating this solve the original equation.

16. $\dfrac{\partial z}{\partial x} = 2xyz.$ $u = \ln z,$ $\dfrac{\partial u}{\partial x} = 2xy.$

17. $z\dfrac{\partial z}{\partial x} = x - y.$ $u = z^2,$ $\dfrac{\partial u}{\partial x} = 2x - 2y.$

18. $\dfrac{\partial z}{\partial x} = ye^z.$ $u = e^{-z},$ $\dfrac{\partial u}{\partial x} = -y.$

Each of the following equations in $u(x,y)$, $u(x,t)$, or $u(x,p)$ involves no differentiation with respect to the second independent variable y, t, or p. Hence if this is treated as a constant, the equation becomes essentially an ordinary differential equation in u and x. Noting that this ordinary equation is linear in u, with constant coefficients, solve each given partial differential equation.

19. $\dfrac{\partial u}{\partial x} - yu = 2x.$ **20.** $\dfrac{\partial u}{\partial x} - u = 2t.$

21. $\dfrac{\partial u}{\partial x} + 2pu = 0.$ **22.** $\dfrac{\partial u}{\partial x} + (4 + 2p)u = 0.$

23. $\dfrac{\partial^2 u}{\partial x^2} - 4y^2 u = 0.$ **24.** $\dfrac{\partial^2 u}{\partial x^2} + 4t^2 u = 0.$

25. $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{p^2}{v^2}\, u = 0.$ **26.** $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{p^2}{v^2}\, u = 0.$

27. Solve the partial differential equation

$$y \frac{\partial^2 z}{\partial x\,\partial y} + \frac{\partial z}{\partial x} = 9x^2 y^2$$

by putting $\frac{\partial z}{\partial x} = p$ and thus reducing the equation to the form

$$y \frac{\partial p}{\partial y} + p = 9x^2 y^2.$$

If x is treated as a constant, this is essentially an ordinary differential equation in y and p.

Show that the system of simultaneous partial differential equations

28. $\frac{\partial z}{\partial x} = 4x - 5y$, $\frac{\partial z}{\partial y} = -5x + 2y$ has as its solution
$z = 2x^2 - 5xy + y^2 + c$.

29. $\frac{\partial z}{\partial x} = y$, $\frac{\partial z}{\partial y} = 2x$ has no solution.

30. $\frac{\partial z}{\partial x} = \cos 2y$, $\frac{\partial z}{\partial y} = -2x \sin 2y$ has as its solution
$z = x \cos 2y + c$.

31. $\frac{\partial z}{\partial x} = 3x^2 y - y^3$, $\frac{\partial z}{\partial y} = x^3 - 3xy^2$ has as its solution
$z = x^3 y - y^3 x + c$.

32. $\frac{\partial^2 z}{\partial x^2} = 0$, $\frac{\partial^2 z}{\partial x\,\partial y} = 0$ has as its solution $z = g(y) + cx$.

33. $\frac{\partial^2 z}{\partial x^2} = 2$, $\frac{\partial^2 z}{\partial x\,\partial y} = 3$, $\frac{\partial^2 z}{\partial y^2} = 4$ has as its solution
$z = x^2 + 3xy + 2y^2 + c_1 x + c_2 y + c_3$.

34. $\frac{\partial^2 z}{\partial x^2} = 4e^{y-2x}$, $\frac{\partial z}{\partial y} = e^{y-2x}$ has as its solution
$z = e^{y-2x} + c_1 x + c_2$.

28. Elimination of Functions

If a given relation between x, y, and z with one or more arbitrary functions is the solution of some partial differential equation,

then that equation may be found by differentiation and elimination. Thus from

$$z = f(x^2 + y^2) \tag{29}$$

we find that

$$\frac{\partial z}{\partial x} = 2xf'(x^2 + y^2), \qquad \frac{\partial z}{\partial y} = 2yf'(x^2 + y^2). \tag{30}$$

Here $f'(u)$ means the derivative of $f(u)$ with respect to u, so that if the arbitrary function happened in a particular case to be $f(u) = \sin u$, we would have $f'(u) = \cos u$.

To find the equation satisfied by the z of Eq. (29), with f the arbitrary function, we must eliminate f and f' from Eqs. (29) and (30). In this case it will suffice to multiply $\frac{\partial z}{\partial x}$ by y, $\frac{\partial z}{\partial y}$ by $-x$, add and deduce from Eq. (30) that

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0. \tag{31}$$

When there is just one arbitrary function, f, differentiation with respect to x and y will lead to two equations containing f and f', in the general case. Then from the three equations f and f' can be eliminated to obtain a single first-order partial differential equation of which the given relation is the solution. A method which sometimes enables us to deduce a solution of this type when the partial differential equation is given is described in Probs. 13 and 17 of Exercise XIV.

When there is more than one arbitrary function, we could take higher derivatives. But it is only exceptionally that at any stage we have the right number of functions and equations. For example, if we start with a relation containing two functions f and g, the first derivatives, with the relation, lead to three equations in f, g, f', and g'. And if we take the three second derivatives we have six equations in six quantities. At the next stage we have ten equations in eight quantities, which is one too many. Hence the given relation in f and g will, in general, not be the solution of any single equation but the common part of the solutions of two third-order equations.

To illustrate the exceptional case when elimination is possible and leads to a single equation of lowest order, consider

$$z = f(2x - y) + g(2x + 3y). \tag{32}$$

Differentiation of this leads to

$$\frac{\partial z}{\partial x} = 2f'(2x - y) + 2g'(2x + 3y)$$

$$\frac{\partial z}{\partial y} = -f'(2x - y) + 3g'(2x + 3y) \tag{33}$$

Here the three equations contain four quantities f, g, f', g'. But by omitting the first one we have two equations in two quantities f', g'. There is still one equation lacking; therefore we find

$$\frac{\partial^2 z}{\partial x^2} = 4f''(2x - y) + 4g''(2x + 3y)$$

$$\frac{\partial^2 z}{\partial x \, \partial y} = -2f''(2x - y) + 6g''(2x - 3y)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(2x - y) + 9g''(2x + 3y). \tag{34}$$

From these three equations in the two quantities f'' and g'', we may eliminate f'' and g''. For example, we may solve the first two equations for f'' and g'' and substitute in the third. The result is

$$3 \frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial^2 z}{\partial x \, \partial y} - 4 \frac{\partial^2 z}{\partial y^2} = 0. \tag{35}$$

EXERCISE XIV

In each case form a first-order partial differential equation by eliminating the arbitrary function f.

1. $z = f(2x - 3y)$.   **2.** $z = f(x)$.   **3.** $z = f(y) + 2xy$.
4. $z = f(xy)$.   **5.** $z = yf(x + 2y)$.   **6.** $z = xf(y/x)$.

In each case form a second-order partial differential equation by eliminating the arbitrary functions f and g.

7. $z = f(3x + 2y) + g(x - y)$.   **8.** $z = f(x) + g(y) + y \ln x$.

9. $z = f(x) + yg(x) + y^2$.   **10.** $z = f(y) + g(2x - y)$.

11. $z = f(x - y) + g(x + y)$.   **12.** $z = f(x) \cdot g(y)$.

13. In some cases the solution of the equation

$$A \frac{\partial z}{\partial x} + B \frac{\partial z}{\partial y} = C, \qquad (a)$$

where A, B, C are given functions of x, y, z, may be found by setting up the system of ordinary differential equations

$$\frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}, \qquad (b)$$

finding two first integrals of this system in the form

$$U(x,y,z) = c_1, \qquad V(x,y,z) = c_2, \qquad (c)$$

with c_1 and c_2 arbitrary constants, and setting

$$U(x,y,z) = f[V(x,y,z)], \qquad (d)$$

with f the arbitrary function.

If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors along the axes, the vector

$$\mathbf{N} = \frac{\partial z}{\partial x}\mathbf{i} + \frac{\partial z}{\partial y}\mathbf{j} - \mathbf{k}$$

is normal to the surface $z = z(x,y)$. Since Eq. (a) expresses that $\mathbf{N} \cdot \mathbf{T} = 0$, where $\mathbf{T} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, $\mathbf{T}$ is perpendicular to $\mathbf{N}$, and thus must lie in the tangent plane of any surface which is a solution of Eq. (a). The system (b) defines curves always tangent to $\mathbf{T}$. And for any value of c_1 or c_2 the surfaces, Eq. (c), are each tangent to $\mathbf{T}$. For a particular f, with each c_2, Eq. (d) matches a $c_1 = f(c_2)$. The one pair c_1, c_2 determine a curve of intersection of surfaces, Eq. (c), and hence tangent to $\mathbf{T}$, which lies on the surface, Eq. (d). As c_2 varies, the curve of intersection sweeps out the whole surface, Eq. (d), which accordingly at each of its points is tangent to $\mathbf{T}$ for that point, and therefore is a solution.

Apply this method to the particular equation $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = z$,

obtaining successively $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$, $\ln x + \ln c_1 = \ln z$,

$\ln x + \ln c_2 = \ln y$, or $z/x = c_1$, $y/x = c_2$, $z = xf(y/x)$. Compare Prob. 6 and its solution.

Use the method of Prob. 13 to show that the solution of

14. $4 \dfrac{\partial z}{\partial x} + 3 \dfrac{\partial z}{\partial y} = 8$ is $z = 2x + f(3x - 4y)$.

15. $a \dfrac{\partial z}{\partial x} + b \dfrac{\partial z}{\partial y} = c$ is $az = cx + f(bx - ay)$.

16. $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = z$ is $z = e^x f(x - y)$.

17. If one or two of the functions A, B, C in Prob. 13 is zero, we understand Eq. (b) of Prob. 13 to mean that the corresponding numerator is zero. Apply this to Eq. (31) of the text to obtain successively $\dfrac{dx}{y} = \dfrac{dy}{-x} = \dfrac{dz}{0}$, $dz = 0$, $z = c_1$, $x\,dx + y\,dy = 0$, $x^2 + y^2 = c_2$, $z = f(x^2 + y^2)$ in agreement with Eq. (29).

Use the method of Prob. 13, as modified in Prob. 17, to show that the solution of

18. $5 \dfrac{\partial z}{\partial x} - 7 \dfrac{\partial z}{\partial y} = 0$ is $z = f(7x + 5y)$.

19. $a \dfrac{\partial z}{\partial x} + b \dfrac{\partial z}{\partial y} = 0$ is $z = f(bx - ay)$.

20. $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 0$ is $z = f\left(\dfrac{y}{x}\right)$.

21. $\dfrac{\partial z}{\partial x} = 0$ is $z = f(y)$, and compare Prob. 1 of Exercise XIII.

22. $2x \dfrac{\partial z}{\partial x} - 3y \dfrac{\partial z}{\partial y} = 0$ is $z = f(x^3 y^2)$.

29. Linear Equations

We shall describe a method of solving partial differential equations of the type

$$3\,\frac{\partial^2 z}{\partial x^2} + 4\,\frac{\partial^2 z}{\partial x\,\partial y} - 4\,\frac{\partial^2 z}{\partial y^2} = 0, \tag{36}$$

made up of terms each of which is a partial derivative of $z(x,y)$ to the same order, here the second, multiplied by some constant coefficient. Such equations are *linear*, or have a first-degree character in z. That is, the result of substituting $c_1 z_1 + c_2 z_2$ in the left number, where c_1 and c_2 are any constants, is the result for z_1, times c_1, plus the result for z_2, times c_2. Hence in particular if z_1 and z_2 are both solutions, they make the left member zero, and so does $c_1 z_1 + c_2 z_2$, which is again a solution.

Let us replace the x derivatives by powers of x, and the y derivatives by powers of y, to obtain

$$3x^2 + 4xy - 4y^2 = 0. \tag{37}$$

Solving this as a quadratic equation in x gives the two roots

$$x = -2y \qquad \text{or} \qquad x = \tfrac{2}{3}y \tag{38}$$

which lead to the factorization

$$3x^2 + 4xy - 4y^2 = 3(x + 2y)\,(x - \tfrac{2}{3}\,y)$$
$$= (x + 2y)(3x - 2y). \tag{39}$$

This shows that Eq. (36) may be written as

$$\left(\frac{\partial}{\partial x} + 2\,\frac{\partial}{\partial y}\right)\left(3\,\frac{\partial}{\partial x} - 2\,\frac{\partial}{\partial y}\right) z = 0. \tag{40}$$

Hence any solution of either first-order equation

$$\frac{\partial z}{\partial x} + 2\,\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad 3\,\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0 \tag{41}$$

will solve Eqs. (40) or (36). For a solution of the second equation will lead to zero when we use the right-hand parenthesis in Eq.

(40), and hence will give zero when we let the operation indicated by the first parenthesis act on this zero.   To show that the same is true of the first equation, we have merely to interchange the order of the factors in Eqs. (39) and (40).

In Prob. 19 of Exercise XIV we showed that the equation

$$a\,\frac{\partial z}{\partial x} + b\,\frac{\partial z}{\partial y} = 0 \qquad \text{had} \qquad z = f(bx - ay) \qquad (42)$$

as its solution.   Hence $z_1 = f(2x - y)$ is a solution of the first equation of Eq. (41), and $z_2 = g(2x + 3y)$ is a solution of the second.   We have written $2x + 3y$ as it is simpler than $-2x - 3y$ and leads to an equivalent result, since an arbitrary function of u is also a function of $-u$, or of ku for any constant $k \neq 0$.

From the linear character of Eq. (36), and the fact that the z_1 and z_2 just found are solutions, $z = z_1 + z_2$ is also a solution.   Thus

$$z = f(2x - y) + g(2x + 3y) \qquad (43)$$

is a solution of Eq. (36).   It is the most general solution because it contains two independent terms containing arbitrary functions, and we started with a second-order equation of special type.   We have used $z_1 + z_2$ instead of $c_1 z_1 + c_2 z_2$, since the last form is equivalent to the first, with different arbitrary functions $c_1 f$ in place of f and $c_2 g$ in place of g.

We note that our result is in accord with the derivation of Eq. (35) as that which had its solution given by Eq. (32).

When there are equal factors, the above process does not give two independent terms.   In this case the general solution is obtained by adding the terms after multiplying one of them by a first-degree factor.   Thus consider

$$\frac{\partial^2 z}{\partial x^2} - 4\,\frac{\partial^2 z}{\partial x\,\partial y} + 4\,\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial}{\partial x} - 2\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - 2\,\frac{\partial}{\partial y}\right) z = 0. \quad (44)$$

Here $z_1 = f(2x + y)$ is a solution, since it leads to zero when acted on by the right-hand parenthesis.   But $z_2 = xg(2x + y)$ is also a solution, because the effect of the right-hand parenthesis on this is

to annihilate the terms in g' and to give the result of acting on x times g, here 1 times g. And the left-hand parenthesis annihilates this. Thus

$$z = f(2x + y) + xg(2x + y) \tag{45}$$

is the general solution of Eq. (44). In place of the x which multiplies g we could use any first-degree factor in x and y, except a multiple of $2x + y$, since this last would not lead to two independent terms. Usually we use x, unless it multiplies $g(x)$, when we use y to give the simplest solution.

Let us next consider equations with left member of the same special type, but with right member a function of x and y. For example,

$$3 \frac{\partial^2 z}{\partial x^2} + 4 \frac{\partial^2 z}{\partial x \, \partial y} - 4 \frac{\partial^2 z}{\partial y^2} = 16 \sin 2y + 16xy. \tag{46}$$

Since the first term contains y alone, we may find a particular solution for it by using $z = Y(y)$ in the left member. Then

$$-4 \frac{d^2 Y}{dy^2} = 16 \sin 2y, \qquad \frac{dY}{dy} = 2 \cos 2y, \qquad Y = \sin 2y, \tag{47}$$

where we have integrated repeatedly and omitted constants of integration.

We could find a particular solution for $16xy$, a polynomial term, by trying a general polynomial of the fourth degree, and using any set of coefficients which would give $16xy$. But in this case we may also note that the middle term would convert Ax^2y^2 into $4Axy$, and hence $4x^2y^2$ into $16xy$. The other terms would lead to $24y^2$, a function of y alone produced by $-y^4/2$ and to $32x^2$, a function of x alone produced by $8x^4/9$. Thus

$$- \tfrac{8}{9}x^4 + 4x^2y^2 + \tfrac{1}{2}y^4 \text{ leads to } 16xy. \tag{48}$$

We next replace the right member by zero, and solve the resulting Eq. (36), obtaining the solution (43). Finally we combine this with the particular solutions in Eqs. (47) and (48). The result is

$$z = \sin 2y - \tfrac{3}{8}x^4 + 4x^2y^2 + \tfrac{1}{2}y^4 + f(2x - y)$$
$$+ g(2x + 3y). \quad (49)$$

Since the first two terms produce the right member and the last two terms produce zero, from its linear character Eq. (46) admits this as a solution. But for this special type of equation, when the order is 2, any solution with two independent terms containing arbitrary functions is the general solution. For a proof of this, and a more general method of solution compare Prob. 12 of Exercise XV.

EXERCISE XV

Solve each of the following partial differential equations:

1. $\dfrac{\partial^2 z}{\partial x^2} - 6\dfrac{\partial^2 z}{\partial x\,\partial y} + 5\dfrac{\partial^2 z}{\partial y^2} = 0.$ **2.** $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x\,\partial y} = 0.$

3. $2\dfrac{\partial^2 z}{\partial x^2} + 5\dfrac{\partial^2 z}{\partial x\,\partial y} + 3\dfrac{\partial^2 z}{\partial y^2} = 0.$ **4.** $3\dfrac{\partial^2 z}{\partial x\,\partial y} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

5. $\dfrac{\partial^2 z}{\partial x^2} - 6\dfrac{\partial^2 z}{\partial x\,\partial y} + 9\dfrac{\partial^2 z}{\partial y^2} = 0.$ **6.** $9\dfrac{\partial^2 z}{\partial x^2} + 4\dfrac{\partial^2 z}{\partial y^2} = 0.$

7. $\dfrac{\partial^2 z}{\partial x\,\partial y} - 4\dfrac{\partial^2 z}{\partial y^2} = 16y.$ **8.** $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial y^2} = 12x^2.$

9. Show that the general solution of Laplace's equation $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = 0$ is $U = f(x + iy) + g(x - iy)$, where $i^2 = -1$.

10. Show that if v is constant and $v \neq 0$, the general solution of the wave equation

$$v^2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{is} \quad u = f(x - vt) + g(x + vt).$$

11. By the method of the text, show that if $p \neq q$ the solution of

$$\frac{\partial^2 z}{\partial x^2} - (p + q)\frac{\partial^2 z}{\partial x\,\partial y} + pq\frac{\partial^2 z}{\partial z^2} = 0$$

or

$$\left(\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - q\frac{\partial}{\partial y}\right)z = 0$$

is $z = f(px + y) + g(qx + y)$. Now put $u = px + y, v = qx + y,$

verify that

$$\frac{\partial}{\partial x} = p\frac{\partial}{\partial u} + q\frac{\partial}{\partial v}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial x} + \frac{\partial}{\partial y},$$

so that

$$\frac{\partial}{\partial x} - p\frac{\partial}{\partial y} = (q - p)\frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial x} - q\frac{\partial}{\partial y} = (p - q)\frac{\partial}{\partial v}.$$

Hence from the factored form the given equation is equivalent to $-(p - q)^2\dfrac{\partial^2 z}{\partial v\,\partial u} = 0$, or $\dfrac{\partial^2 z}{\partial v\,\partial u} = 0$. This may be solved by successive integration, as in Sec. 26, to give $z = f(u) + g(v)$ or $z = f(px + y) + g(qx + y)$, which proves that this is the general solution.

12. We may devise a systematic method of solving equations with a right member by the substitution of Prob. 11. Thus to solve $\dfrac{\partial^2 z}{\partial x^2} - (p + q)\dfrac{\partial^2 z}{\partial x\,\partial y} + pq\dfrac{\partial^2 z}{\partial y^2} = F(x,y)$ where $F(x,y)$ is any given function and $p \neq q$, we put $u = px + y$, $v = qx + y$. Using Prob. 11 for the left member, show that the equation becomes $-(p - q)^2\dfrac{\partial^2 z}{\partial v\,\partial u} = F\left(\dfrac{u - v}{p - q}, \dfrac{pv - qu}{p - q}\right)$, which may be solved as in Sec. 26 to give $z =$ a particular integral $+ f(u) + g(v)$. This process is usually longer than the tentative method used in the text for Eq. (46), but it proves that the complete solution of an equation of this type contains just two arbitrary functions when $p \neq q$.

13. If we let $q = p$ in the equation of Prob. 12, it becomes $\dfrac{\partial^2 z}{\partial x^2} - 2p\dfrac{\partial^2 z}{\partial x\,\partial y} + p^2\dfrac{\partial^2 z}{\partial y^2} = F(x,y)$. If $p = 0$, this may be solved by successive integration as it stands. When $p \neq 0$, show that we may solve by a procedure similar to that of Prob. 12, if we use the substitution $u = px + y$, $v = y$.

30. Particular Solutions

The solution of a particular physical problem cannot contain any constants or functions which may be given arbitrary values.

If one of the conditions in the mathematical formulation of a problem is a differential equation, there must be auxiliary relations at hand which serve to determine the arbitrary elements in the solution of the differential equation. Sometimes we obtain the specific solution desired by first finding the general solution, including the arbitrary elements, and then determining these elements from the initial conditions, or boundary values. Although this is the usual procedure for elementary problems involving ordinary differential equations, where arbitrary constants appear, it is less often applicable to problems involving partial differential equations because of the difficulty of fitting arbitrary functions to the auxiliary conditions.

An alternative method is to find particular solutions of the partial differential equation which satisfy some of the boundary conditions and then to combine these particular solutions in such a way that all the conditions of the physical problem are met. In this connection, solutions equal to a product of factors, each factor being a function of only one of the independent variables, are useful. Accordingly we shall outline a systematic procedure for finding such particular solutions.

As an example to which our method applies, let us consider Laplace's equation in two variables

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \tag{50}$$

We wish to find all solutions, if there are any, of the form

$$U = X(x) \cdot Y(y), \qquad \text{or} \qquad U = XY. \tag{51}$$

In the second form we have written X for $X(x)$ and Y for $Y(y)$, relying on our notation to remind us that X is a function of x alone and Y is a function of y alone. We also use primes for ordinary derivatives so that

$$X' = X'(x) = \frac{dX}{dx} \qquad \text{and} \qquad X'' = X''(x) = \frac{d^2X}{dx^2}, \tag{52}$$

with similar defining relations for Y' and Y''. Then by succes-

sive partial differentiation we find from Eq. (51) that

$$\frac{\partial U}{\partial x} = X'Y, \qquad \frac{\partial^2 U}{\partial x^2} = X''Y,$$

$$\frac{\partial U}{\partial y} = XY', \qquad \frac{\partial^2 U}{\partial y^2} = XY''. \tag{53}$$

Now insert these values in Eq. (50).   The result,

$$X''Y + XY'' = 0 \qquad \text{or} \qquad X''Y = -XY'', \tag{54}$$

is a condition that the U of Eq. (51) be a solution of the given partial differential equation.   We may separate the variables in the last equation, and write it in the form

$$\frac{X''}{X} = -\frac{Y''}{Y}. \tag{55}$$

Such separation of the variables is not possible for all equations. Our procedure succeeds only when the equation in X, Y and their derivatives is capable of separation.

We now observe that the left member of Eq. (55) does not involve y, and hence cannot change when y changes while x is kept fixed.   Similarly the right member does not involve x, and hence cannot change when x changes while y is kept fixed.   As the two members are equal, their common value does not change when we change the variables one at a time.   Since we may go from any values x_1, y_1 to any other values x_2, y_2 by first changing x_1 to x_2, with $y = y_1$ and then changing y_1 to y_2, with $x = x_2$, the common value is the same at x_2, y_2 as at x_1, y_1 and therefore must be a constant k.   Thus we may write

$$\frac{X''}{X} = -\frac{Y''}{Y} = k, \qquad \text{or} \qquad X'' = kX, \qquad Y'' = -kY. \tag{56}$$

In more familiar notation, the last two equations are

$$\frac{d^2X}{dx^2} - kX = 0, \qquad \frac{d^2Y}{dy^2} + kY = 0. \tag{57}$$

The form of solution of these linear differential equations

with constant coefficients depends on the roots of the equations $m^2 - k = 0$ and $m^2 + k = 0$. First let k be negative. Then $-k$ is positive, and the roots involve $\sqrt{-k}$. To simplify the writing, put $\sqrt{-k} = a$, or $k = -a^2$. Then the equation $m^2 + a^2 = 0$ has roots ai and $-ai$. And these lead to the solution

$$X = c_1 \sin ax + c_2 \cos ax. \tag{58}$$

The equation $m^2 - a^2$ has roots a and $-a$, which lead to

$$Y = c_3 e^{ay} + c_4 e^{-ay}. \tag{59}$$

Next let $k = 0$. Then each of the two equations in m reduces to $m^2 = 0$, with roots $0,0$ in each case. These lead to

$$X = c_5 + c_2 x \quad \text{and} \quad Y = c_7 + c_8 y. \tag{60}$$

Finally let k be positive. Here the roots involve $\sqrt{k}$, and to simplify the writing we put $\sqrt{k} = b$, or $k = b^2$. The first equation is now $m^2 - b^2 = 0$, with roots b and $-b$. These lead to

$$X = c_9 e^{bx} + c_{10} e^{-bx}. \tag{61}$$

The equation $m^2 + b^2 = 0$, with roots bi and $-bi$, leads to

$$Y = c_{11} \sin by + c_{12} \cos by. \tag{62}$$

From Eqs. (58) to (62), and Eq. (51) we find as the desired particular solutions of the given equation, Eq. (50)

$$\begin{aligned} U &= (c_1 \sin ax + c_2 \cos ax)(c_3 e^{ay} + c_4 e^{-ay}), \\ U &= (c_5 + c_6 x)(c_7 + c_8 y), \\ U &= (c_9 e^{bx} + c_{10} e^{-bx})(c_{11} \sin by + c_{12} \cos by). \end{aligned} \tag{63}$$

Each of these solutions apparently contains four constants c, but really only three independent ones, since one of them may be divided out. For example, in the first form, if $c_1 \neq 0$, we may write

$$\begin{aligned} U &= \left(\sin ax + \frac{c_2}{c_1} \cos ax \right)(c_1 c_3 e^{ay} + c_1 c_4 e^{-ay}) \\ &= (\sin ax + c_2{}' \cos ax)(c_3{}' e^{by} + c_4{}' e^{-by}). \end{aligned} \tag{64}$$

The advantage of not replacing any of the constants in the solutions (63) by unity as we have just done with c_1, in effect, in obtaining Eq. (64) is that we do not exclude the possibility of any of them being zero.

Since Eq. (50) has a linear character, the sum of any number of solutions is again a solution. Hence if we take any number of solutions of one or more of the forms given in Eq. (63), obtained by giving different values to a, b, and the constants c, and add the results we will have a solution of Eq. (50). We may even combine an infinite number of such terms into an infinite series, provided the series converges in such a way that it may be differentiated termwise twice.

Let us next consider the first-order equation

$$2x \frac{\partial z}{\partial x} - 3y \frac{\partial z}{\partial y} = 0. \tag{65}$$

This will have a solution of the form given in Eq. (51), or

$$z = XY, \tag{66}$$

provided that

$$2xX'Y - 3yXY' = 0, \qquad \text{or} \qquad 2xX'Y = 3yXY'. \tag{67}$$

We may separate the variables by writing this in the form

$$2x \frac{X'}{X} = 3y \frac{Y'}{Y}. \tag{68}$$

The argument used for Eq. (55) shows that each member of this equation must equal a constant k. Consequently,

$$2x \frac{X'}{X} = 3y \frac{Y'}{Y} = k, \qquad \text{or} \qquad 2xX' = kX, \qquad 3yY' = kY. \tag{69}$$

The equation in X, in more familiar notation, is

$$2x \frac{dX}{dx} = kX, \qquad \text{or} \qquad \frac{dX}{X} = \frac{k}{2} \frac{dx}{x}. \tag{70}$$

This has as its integral

$$\ln X = \frac{k}{2} \ln x + c_1, \qquad \text{so that} \qquad X = e^{c_1} x^{k/2}. \tag{71}$$

The equation in Y is

$$3y \frac{dY}{dy} = kY, \qquad \text{and has} \qquad Y = e^{c_2} y^{k/3} \tag{72}$$

as its solution.   From Eqs. (71), (72), and (66) we find that

$$z = XY = e^{c_1} e^{c_2} x^{k/2} y^{k/3}. \tag{73}$$

This may be simplified by putting $e^{c_1} e^{c_2} = c$, a new constant, and setting $k = 6a$ to avoid fractions.   Thus

$$z = cx^{3a} y^{2a} \tag{74}$$

is the solution of the given equation of the desired form.

To see how a fairly general solution may be built up out of particular solutions, we recall that in Prob. 22 of Exercise XIV the general solution of Eq. (65) was found to be

$$z = f(x^3 y^2). \tag{75}$$

Suppose that $f(u)$ is a regular analytic function for $u = 0$, so that it admits of a Maclaurin expansion

$$f(u) = A_0 + A_1 u + A_2 u^2 + \cdots + A_n u^n + \cdots. \tag{76}$$

Then the solution (75) admits the expansion

$$z = A_0 + A_1 x^3 y^2 + A_2 (x^3 y^2)^2 + \cdots + A_n (x^3 y^2)^n + \cdots, \tag{77}$$

which is an infinite series of terms each of the form (74), with a equal to 0, 1, 2, 3, $\cdots$, n, $\cdots$ and $c = A_n$ for $a = n$.

EXERCISE XVI

1. The equation for one-dimensional heat flow or diffusion was found in Eq. (5) to be

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}$$

Show that the only solutions of this equation of the special form $U = X(x) \cdot T(t)$ are of one of the three forms:

$$U = (c_1 e^{kx} + c_2 e^{-kx}) e^{a^2 k^2 t},$$
$$U = (c_3 \sin bx + c_4 \cos bx) e^{-a^2 b^2 t},$$
$$U = c_5 + c_6 x.$$

Find particular solutions of the form $X(x) \cdot Y(y)$ for each of the following partial differential equations:

2. $2 \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0.$ **3.** $y \dfrac{\partial z}{\partial x} - x \dfrac{\partial z}{\partial y} = 0.$ **4.** $x \dfrac{\partial z}{\partial x} + y \dfrac{\partial z}{\partial y} = 0.$

5. $e^y \dfrac{\partial z}{\partial x} = e^x \dfrac{\partial z}{\partial y}.$ **6.** $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 2z.$ **7.** $y^2 \dfrac{\partial z}{\partial x} + x^2 \dfrac{\partial z}{\partial y} = 0.$

8. $\dfrac{\partial^2 z}{\partial x^2} = 4 \dfrac{\partial^2 z}{\partial y^2}$ **9.** $\dfrac{\partial^2 z}{\partial x\, \partial y} = 2 \dfrac{\partial^2 z}{\partial y^2}.$ **10.** $\dfrac{\partial^2 z}{\partial x^2} = -4 \dfrac{\partial^2 z}{\partial y^2}.$

Find particular solutions of the form $X(x) \cdot T(t)$ for each of the following equations:

11. $\dfrac{\partial^2 U}{\partial x^2} = 9 \dfrac{\partial^2 U}{\partial t^2}.$ **12.** $\dfrac{\partial^2 U}{\partial x^2} = -9 \dfrac{\partial^2 U}{\partial t^2}.$ **13.** $\dfrac{\partial^2 U}{\partial x^2} = 3 \dfrac{\partial^2 U}{\partial x\, \partial t}.$

14. Show that the differential equation

$$r^2 \frac{\partial^2 U}{\partial r^2} + r \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \theta^2} = 0$$

has particular solutions of the form:

$$U = (c_1 r^k + c_2 r^{-k})(c_3 \sin k\theta + c_4 \cos k\theta).$$

15. Show that the differential equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r} \frac{\partial U}{\partial r} + a^2 U = 0$$

has particular solutions of the form:

$$U = [c_1 J_n(ar) + c_2 Y_n(ar)](c_3 \sin n\theta + c_4 \cos n\theta),$$

where $J_n(x)$ and $Y_n(x)$ are two independent solutions of the

ordinary differential equation known as *Bessel's equation*:

$$\frac{d^2X}{dx^2} + \frac{1}{x}\frac{dX}{dx} + \left(1 - \frac{n^2}{x^2}\right)X = 0.$$

16. For $n = 0$ or a positive integer, *Bessel's function* of order n is defined by

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{n+2k}}{2^{n+2k} k!(n+k)!} \qquad \text{with } 0! = 1 \text{ in the first term.}$$

By termwise substitution, verify that this is a solution of Bessel's equation of order n, given at the end of Prob. 15.

17. From Eq. (7) of Sec. 2, deduce that

$$e^{xt/2} = \sum_{r=0}^{\infty} \frac{x^r}{2^r}\frac{1}{r!} t^r \qquad \text{and} \qquad e^{-\frac{x}{2}\frac{1}{t}} = \sum_{s=0}^{\infty} (-1)^s \frac{x^s}{2^s}\frac{1}{s!} t^{-s}.$$

18. Let $J_{-n}(x) = (-1)^n J_n(x)$ for n zero or a positive integer. Multiply the series in Prob. 17 termwise, and by using the expansion of Prob. 16 show that $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$.

19. From Prob. 18, with $t = e^{i\phi}$, deduce that

$$e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)e^{in\phi}.$$

20. By taking real and imaginary components of the equation of Prob. 19, deduce the expansions

$$\cos(x\sin\phi) = J_0(x) = 2J_2(x)\cos 2\phi$$
$$+ 2J_4(x)\cos 4\phi + \cdots,$$
$$\sin(x\sin\phi) = 2J_1(x)\sin\phi + 2J_3(x)\sin 3\phi$$
$$+ 2J_5(x)\sin 5\phi + \cdots.$$

21. The equation of Prob. 19 is a complex Fourier series for the complex function of the real variable ϕ of period 2π, $e^{ix\sin\phi}$. Apply Eq. (95) of Sec. 21 with c, p, x, ω replaced by $-\pi$, 2π,

ϕ, 1, respectively, to show that

$$J_n(x) = \frac{1}{\pi} \int_0^\pi \cos\ (x \sin\ \phi - n\phi)d\phi.$$

31. Vibrations. Wave Equations

Let us consider a tightly stretched string, vibrating in a plane. Call the weight per unit of length D (lb./ft.) and the tension T (lb.). We take x (ft.) as the coordinate along the equilibrium position of the string, the x axis in Fig. 39. And u (ft.) is the
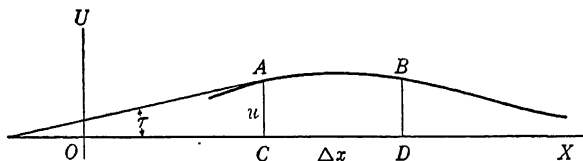
FIG. 39.

distance from a point on the string to its corresponding equilibrium position. We are thinking of small vibrations, and therefore neglect the small motion parallel to the x axis due to the displaced position of the string not being a straight line. Thus the mass of the displaced segment AB is taken as that of the segment CD, or $(D/g)\Delta x$. And for this segment AB, the product of mass times acceleration parallel to the x axis is

$$\frac{D}{g}\ \Delta x\ \frac{\partial^2 u}{\partial t^2}\ \bigg|_{x'}, \tag{78}$$

where x' is a suitably chosen value between x and $x + \Delta x$.

We assume that the tension is so large compared with the weight that the effect of gravity is negligible. Then the effective forces on the segment are due to the tension at the ends. These are along the tangent to the curve giving the position of the string, whose inclination τ to the x axis is small. Thus we may neglect the difference between $\sin \tau$ and $\tan \tau = \dfrac{\partial u}{dx}$. Hence the u component of tension at A is $-T \sin \tau$, or with our approxima-

tion, $-T \tan \tau = -T \frac{\partial u}{\partial x}\Big|_x$. Similarly for B we find $T \frac{\partial u}{dx}\Big|_{x+\Delta x}$. And the resultant force parallel to the u axis is

$$T \left(\frac{\partial u}{\partial x}\Big|_{x+\Delta x} - \frac{\partial u}{\partial x}\Big|_x\right). \tag{79}$$

But force equals mass times acceleration. Hence we may equate the expressions for their u components given in Eqs. (79) and (78). If we do this, divide through by Δx, and take the limit as $\Delta x \to 0$, $x + \Delta x$ and the intermediate value x' will both approach x. Hence we have in the limit

$$\frac{D}{g} \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2}. \tag{80}$$

This is the equation for a vibrating string, valid when $\left(\frac{\partial u}{\partial x}\right)^2$ is small compared with unity, since the various approximations made all amount to a replacement of the factor

$$\sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2} = \cos \tau$$

by unity in certain places.

If we replace gT/D by v^2 (ft.²/sec.²), Eq. (80) becomes

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}, \quad \text{or} \quad \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}. \tag{81}$$

The solution of this equation was found in Prob. 10 of Exercise XV to be $u = f(x - vt) + g(x + vt)$. If we calculate $x - vt$ for a particular value x_1 at time t_1, and then for a time t_0 sec. later but at a position vt_0 to the right, we get the same value. For if

$$t_2 = t_1 + t_0, \quad x_2 = x_1 + vt_0, \quad \text{then } x_2 - vt_2 = x_1 - vt_1. \tag{82}$$

This shows that the term $f(x - vt)$ represents a wave traveling to the right with velocity v. Similarly $g(x + vt)$ represents a

wave traveling to the left with velocity v. For this reason, Eq. (81) is called the *wave equation* in one dimension.

We may use any convenient units for u and x, provided that the units of v are the same as those of x/t.

An argument similar to that given for the string may be used for a vibrating stretched membrane such as a drumhead. Since the element is here a small rectangle instead of a small segment, there is a term in $\frac{\partial^2 u}{\partial y^2}$ as well as a term in $\frac{\partial^2 u}{\partial x^2}$. Thus we find

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{83}$$

as the equation for a vibrating membrane when $v^2 = gT/D$, where here T is in pounds per foot and D is in pounds per square foot.

The *wave equation* in three dimensions is

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{84}$$

Here t is time, x, y, z are lengths all with the same unit, and v is a velocity having the same units as x/t. For sound waves in a solid or in air, u is a displacement. An equation of this form also holds for electromagnetic waves in empty space. In this case u is one of the three components of the electric field intensity vector, or of the magnetic field intensity vector. See Sec. 36 and in particular Eq. (117).

32. Curvilinear Coordinates

For problems in a plane involving circular symmetry about the origin, it is often convenient to introduce polar coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{85}$$

In most discussions of partial differentiation, it is shown as an illustration or application of the theory that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r}\frac{\partial U}{\partial r} \tag{86}$$

This identity holds whether U is a function of two variables only, first x,y and then r,θ or of these pairs together with other variables such as z and t which remain unchanged.   Thus we may use Eq. (86) to transform Laplace's equation, Eq. (8), or the heat equation, Eq. (7), to cylindrical coordinates r, θ, z.   For example, the heat equation, Eq. (7), becomes

$$\frac{\partial U}{\partial t} = a^2 \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} \right). \tag{87}$$

Similarly we find that the wave equation, Eq. (84), becomes

$$\frac{1}{v^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2}. \tag{88}$$

For problems in space involving spherical symmetry about the origin, it is more convenient to introduce spherical coordinates

$$x = r \cos \theta \sin \phi, \qquad y = r \sin \theta \sin \phi, \qquad z = r \cos \phi. \tag{89}$$

Here r is a radial coordinate giving the distance from the origin, θ is an angular coordinate of longitude, and ϕ is a coordinate of colatitude measured down from the z axis.   For these coordinates

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = \frac{\partial^2 U}{\partial r^2} + \frac{2}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2 \sin^2 \phi} \frac{\partial^2 U}{\partial \theta^2}$$
$$+ \frac{1}{r^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\cot \phi}{r^2} \frac{\partial U}{\partial \phi}. \tag{90}$$

This may be derived by a lengthy elementary calculation, or more briefly by methods based on symbolic vector operators or on the calculus of variations.   Any partial differential equation in which the space coordinates x, y, z appear in the Laplacian expression only, for example Eqs. (7) and (84), may be expressed in spherical coordinates by means of Eq. (90).

EXERCISE XVII

1. For one choice of the functions f and g in the general solution of Eq. (81) it becomes $u = B \cos b(x - vt) - B \cos b(x + vt)$.

Show that this may be written $u = 2B \sin bx \sin bvt$, and thus has the form $X(x) \cdot T(t)$ used in Sec. 29.

2. If the string is of length L and is fixed at both ends, we must have $u(0,t) = 0$ and $u(L,t) = 0$. Show that the second condition will be met if we take $b = n\pi x/L$, where n is any positive integer, in the solution of Prob. 1. With $2B = A$, this gives the harmonic vibrations $u = A \sin \dfrac{n\pi x}{L} \sin \dfrac{n\pi vt}{L}$.

3. The fundamental note of the string is given by the solution of Prob. 2 with $n = 1$. Recalling the definition of $v = \sqrt{gT/D}$, deduce that the pitch of the fundamental note of a musical string is $(1/2L)\sqrt{gT/D}$. Thus the pitch is proportional to the square root of the tension, and is inversely proportional to the length and the square root of the density.

4. For a string whose density and tension vary with position, show that the equation is

$$\frac{D(x)}{g} \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[T(x) \frac{\partial u}{\partial x}\right].$$

5. Apply the method of Sec. 30 to Eq. (81), and so deduce that

$$u = (c_1 \sin kx + c_2 \cos kx)(c_3 \sin kvt + c_4 \cos kvt)$$

is a particular solution.

6. For a vibrating string with viscous damping the equation is

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} - b \frac{\partial u}{\partial t}$$

By the method of Sec. 30 deduce that this admits the particular solution

$$u = e^{-bt/2} (c_1 \sin kx + c_2 \cos kx)(c_3 \sin Mt + c_4 \cos Mt)$$

where $M = \sqrt{k^2 v^2 - (b^2/4)}$.

7. Use Eq. (87) to check Prob. 7 of Exercise XII.

8. Use Eqs. (90) and (7) to check Prob. 11 of Exercise XII.

9. Assume that u depends on the spherical r, but not on θ or ϕ.

By combining Eqs. (84) and (90) deduce the equation for spherical waves

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial r}{\partial r}\right).$$

10. The general solution of the equation of Prob. **9** is

$$u = \frac{1}{r}f(r - vt) + \frac{1}{r}g(r + vt).$$

Verify by direct substitution that this satisfies the equation.

11. Check the solution of Prob. 10 by using the substitution $u(r,t) = U(r,t)/r$ to reduce the equation of Prob. 9 to a wave equation in $U(r,t)$ with constant coefficients.

12. From Eqs. (86) and (83) show that in cylindrical coordinates the equation of a vibrating membrane is

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

13. We may obtain an expression for the harmonic vibrations of the membrane of Prob. 12 by putting $u(r,\theta,t) = \sin btU(r,\theta)$. Make this substitution, and show that if $b = av$, the equation in U becomes that for which particular solutions were found in Prob. 15 of Exercise XVI.

33. Transmission of Electricity

Let us study the flow of electricity in a long imperfectly insulated cable. In Fig. 40, AB is the cable, and the current flows from the source through AB, then through the load, and back through $B'A'$ to the source. We shall speak of the return path $A'B'$ as the ground, although it may be a second cable. The arrow indicates that the direction from A to B is considered to be positive in measuring current and difference of potential. Compare Secs. 11 and 13. If x (miles) is the distance along the cable from A, the emf e (volts) and the current i (amperes) will

depend on x as well as on the time t (seconds).   That is,

$$e = e(x,t), \qquad i = i(x,t). \tag{91}$$

The series resistance R (ohms per mile) and series inductance L (henrys per mile) are so defined that for a short segment of the line from x to $x + \Delta x$, $R\,\Delta x$ and $L\,\Delta x$ have meanings for the segment somewhat similar to the lumped resistance and induct-
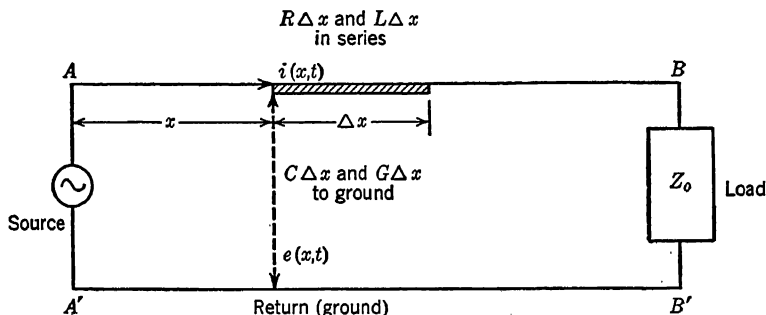
Fig. 40.

ance for an ordinary simple circuit.   And the emf decreases with distance in accord with the relation

$$- \frac{\partial e}{\partial x} = Ri + L\,\frac{\partial i}{\partial t}. \tag{92}$$

The capacitance to ground C (farads per mile) and conductance to ground G (mhos per mile) are so defined that the decrease in current for the short segment is somewhat similar to that in two parallel circuits, one with lumped capacity $C\,\Delta x$ and one with lumped resistance $1/(G\,\Delta x)$.   And the current decreases with distance in accord with the relation

$$- \frac{\partial i}{\partial x} = Ge + C\,\frac{\partial e}{\partial t}. \tag{93}$$

The two equations (92) and (93) together determine e and i in terms of x and t.   They are called the *transmission-line equations*.

In some applications, a different convention as to direction is used, and the equations appear without the minus signs.

We may eliminate i from our equations by differentiating Eq. (92) with respect to x. The resulting equation · contains $\dfrac{\partial i}{\partial x}$ whose value is given by Eq. (93) as it stands, and $\dfrac{\partial^2 i}{\partial x\, \partial t}$ whose value is obtained from Eq. (93) by differentiation with respect to t. Substitution of these values leads to

$$\frac{\partial^2 e}{\partial x^2} + LC\, \frac{\partial^2 e}{\partial t^2} + (RC + LG)\, \frac{\partial e}{\partial t} + RGe, \tag{94}$$

a relation which e must satisfy. By a similar procedure we may eliminate e and its derivatives from Eqs. (92) and (93) and thus derive

$$\frac{\partial^2 i}{\partial x^2} + LC\, \frac{\partial^2 i}{\partial t^2} + (RC + LG)\, \frac{\partial i}{\partial t} + RGi \tag{95}$$

as the relation which i must satisfy.

Equations (92) to (95) not only apply to power lines, but are also used in discussing telephony, telegraphy, and radio antennas. In many applications to telegraph signaling, the leakage is small, and the term for the effect of inductance is negligible, so that we may set $G = 0$ and $L = 0$. Equations (92) to (94) then simplify to

$$-\frac{\partial e}{\partial x} = Ri, \qquad -\frac{\partial i}{\partial x} = C\, \frac{\partial e}{\partial t}, \qquad \frac{\partial^2 e}{\partial x^2} = RC\, \frac{\partial e}{\partial t}. \tag{96}$$

These are known as the *telegraph* or *cable equations.*

When the frequencies are high, the terms in the time derivatives are large, and some qualitative properties of the solution may be found by neglecting the terms for the losses due to leakage and resistance in comparison with them. For this *lossless line*, $G = 0$ and $R = 0$ so that the simplified equations

$$-\frac{\partial e}{\partial x} = L\, \frac{\partial i}{\partial t}, \qquad -\frac{\partial i}{\partial x} = C\, \frac{\partial e}{\partial t}, \qquad \frac{\partial^2 e}{\partial x^2} = LC\, \frac{\partial^2 e}{\partial t^2} \tag{97}$$

apply. They are known as the *radio equations.*

EXERCISE XVIII

1. Show that the general solution of the radio equations, Eqs. (97), may be written

$$e = f(x - vt) + g(x + vt),$$

$$i = \sqrt{\frac{C}{L}}\,[f(x - vt) - g(x + vt)], \quad \text{where } v = \frac{1}{\sqrt{LC}}.$$

2. By the argument used for Eqs. (81) and (82) interpret the solutions of Prob. 1 as a combination of two waves, one moving to the right and the other to the left, each with velocity $v = 1/\sqrt{LC}$.

3. If e and i are functions of x and t, and k is any constant, the equations of transformation $e = e_1\epsilon^{-kt}$, $i = i_1\epsilon^{-kt}$ define e_1 and i_1 as two new functions of x and t. When Eqs. (92) and (93) hold, show that e_1 and i_1 satisfy the equations

$$-\frac{\partial e_1}{\partial x} = (R - kL)i_1 + L\,\frac{\partial i_1}{\partial t},$$

$$-\frac{\partial i_1}{\partial x} = (G - kC)e_1 + C\,\frac{\partial e_1}{\partial t}.$$

4. A transmission line whose constants satisfy the relation $LG = RC$ is called a *distortionless line*. In this case $G/C = R/L$. Show that if we transform variables as in Prob. 3, with $k = R/L$, the equations in e_1 and i_1 have the form of Eqs. (97). Use this fact, and Prob. 1, to find the general solution for the distortionless line in the form

$$e = \epsilon^{-kt}[f(x - vt) + G(x + vt)],$$

$$i = \epsilon^{-kt}\sqrt{\frac{C}{L}}\,[f(x - vt) - g(x + vt)], \quad \text{where } v = \frac{1}{\sqrt{LC}},$$

$$k = \frac{R}{L} = \frac{G}{C}.$$

5. Interpret the solution of the distortionless line found in Prob. 4 as a combination of waves which preserve their shape, but die down exponentially. Compare Prob. 2.

6. Consider a power line whose source or impressed emf is a single sine term. After the steady state has been reached, the current and emf will involve t through a factor of this same frequency. Thus we may write

$$e = E_m(x) \sin [\omega t + \phi_1(x)], \qquad i = I_m(x) \sin [\omega t + \phi_2(x)].$$

We now define *complex* quantities $F(x)$ and $H(x)$ by the relations

$$F(x) = E_m(x) \epsilon^{j\phi_1(x)}, \qquad H(x) = I_m(x) \epsilon^{j\phi_2(x)}.$$

Then, with Im meaning imaginary component as in sec. 12, we have

$$e = \text{Im complex } e = \text{Im } F(x) \epsilon^{j\omega t},$$
$$i = \text{Im complex } i = \text{Im } H(x) \epsilon^{j\omega t},$$

and the complex e and complex i, abbreviated by e and i, satisfy Eqs. (92) and (93). Show that the condition for this is that $F(x)$ and $H(x)$ are solutions of

$$-\frac{dF}{dx} = (R + j\omega L)H, \qquad -\frac{dH}{dx} = (G + j\omega C)F,$$

a system of simultaneous ordinary differential equations.

7. The following notation is useful in discussing the transmission line of Prob. 6. Let $Z = R + j\omega L$ be the complex series impedance and $Y = G + j\omega C$ be the complex shunt admittance. Define the propagation constant $\gamma = \alpha + j\beta = \sqrt{ZY}$, and the characteristic impedance $Z_K = \sqrt{Z/Y}$. Then the Eqs. of Prob. 6 are $-dF/dx = ZH$, $-dH/dx = YF$. Show that these imply $d^2F/dx^2 = ZYF = \gamma^2 F$. Hence find F, and then H from the first equation, and deduce that

$$\text{complex } e = K_1 \epsilon^{\gamma x + j\omega t} + K_2 \epsilon^{-\gamma x + j\omega t},$$
$$\text{complex } i = \frac{\text{complex } e}{Z_k}$$

where $K_1 = A\epsilon^{ja}$, $K_2 = B\epsilon^{jb}$ are complex arbitrary constants.

8. Deduce from Prob. 7 that for the real emf,

$$e = A\epsilon^{\alpha x} \sin (\omega t + \beta x + a) + B\epsilon^{-\alpha x} \sin (\omega t - \beta x + b).$$

9. Show that, when $R = 0$ and $G = 0$, the particular complex solutions found in Prob. 7 are special cases of the general solution of the radio equations found in Prob. 1.

10. If the line is of length x_1 and is terminated in a load impedance Z_0, then for the complex values e, i, $e = iZ_0$ when $x = x_1$. If the real input voltage is $E_i \sin \omega t$, then for the complex e, $e = E_i e^{j\omega t}$ when $x = 0$. Show that these two conditions determine the value of the complex constants K_1, K_2 in Prob. 7.

11. Consider the particular case of Prob. 10 when the line is short-circuited at $x = x_1$, or $Z_0 = 0$. Show that in this case for the real e

$$e = -D[\sinh \alpha(x - x_1) \sin (\omega t - \phi) \cos \beta(x - x_1) \\ + \cosh \alpha(x - x_1) \cos (\omega t - \phi) \sin \beta(x - x_1)],$$

where

$$D = \frac{E_i}{\sqrt{\sinh^2 \alpha x_1 + \sin^2 \alpha x_1}}, \qquad \phi = \tan^{-1} (\tan \beta x_1 \coth \beta x_1),$$

with ϕ in a quadrant such that $\sin \beta x_1$ and $\sin \phi$ have the same sign.

34. Maxwell's Equations

An electromagnetic field may be mathematically characterized by five vectors **E**, **H**, **D**, **B**, and **J**. In terms of the differential operations curl and div (divergence) whose meaning we shall recall in detail later, the differential equations satisfied by these vectors may be written

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t},$$

$$\text{div } \mathbf{B} = 0,$$

$$\text{div } \mathbf{D} = \rho, \tag{98}$$

In a homogeneous isotropic medium we also have the propor-

tionality relations

$$D = KE, \qquad B = \mu H, \qquad J = \sigma E. \qquad (99)$$

Equations (98) and (99) are the fundamental *Maxwell equations* when the MKS rationalized system of units is used.   In this case

$E = E_x i + E_y j + E_z k$ = electric field intensity (volt/meter)
$H = H_x i + H_y j + H_z k$
        = magnetic field intensity (ampere-turn/meter)
$D = D_x i + D_y j + D_z k$ = electric flux density (coulomb/meter^2)
$B = B_x i + B_y j + B_z k$ = magnetic flux density (weber/meter^2)
$J = J_x i + J_y j + J_z k$ = current density (ampere/meter^2)

For each vector the second expression is written in terms of i, j, k the unit vectors along the coordinate axes and the scalar components along these axes, indicated by subscripts.   For a particular point in the region where the field exists, and a particular instant of time, any one of the five vectors has a definite magnitude and direction, and thus is a vector function of position and time.   Thus each component depends on the coordinates xyz of the point at which it is to be calculated and on the time t. Hence a complete description of the electromagnetic field in terms of the scalar components involves fifteen functions of x, y, z, and t.

We confine our attention to the case where Eqs. (99) hold with the coefficients K, μ, σ constant throughout the field under consideration.   Then we may use these relations to eliminate D, B, and J from Eqs. (98) which then take the form:

$$\operatorname{curl} E = -\mu \frac{\partial H}{\partial t},$$

$$\operatorname{curl} H = \sigma E + K \frac{\partial E}{\partial t},$$

$$\operatorname{div} H = 0, \qquad\qquad (100)$$

$$\operatorname{div} E = \frac{\rho}{K}.$$

The first of these relations is equivalent to the three component equations

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t},$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t}, \qquad (101)$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t}.$$

The components of curl **H** are analogous to those just written for curl **E**. Thus, for example, the first component of the second equation is

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \sigma E_x + K \frac{\partial E_x}{\partial t}. \qquad (102)$$

The last two equations, Eqs. (100), when written explicitly in terms of components, become

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0,$$

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{K}. \qquad (103)$$

35. Electrostatic Fields

In a static field the current density $\mathbf{J} = \sigma\mathbf{E}$ is zero, and the other four field vectors are independent of the time. Thus $\frac{\partial H}{\partial t} = 0$ and $\frac{\partial E}{\partial t} = 0$, and Eqs. (100) reduce to

$$\text{curl } \mathbf{E} = 0, \quad \text{curl } \mathbf{H} = 0, \quad \text{div } \mathbf{H} = 0, \quad \text{div } \mathbf{E} = \frac{\rho}{K}. \quad (104)$$

But curl $\mathbf{E} = 0$ is the condition that there exists a V_E such that

$$-\mathbf{E} = \text{grad } V_E = \frac{\partial V_E}{\partial x}\mathbf{i} + \frac{\partial V_E}{\partial y}\mathbf{j} + \frac{\partial V_E}{\partial z}\mathbf{k}, \qquad (105)$$

where the last expression shows the meaning of the differential operation grad (gradient). V_E is called the *electric scalar potential function*. For any function V_E, the **E** formed from it by Eq. (105)

will satisfy the first equation of Eq. (104). And the last will be satisfied if

$$\text{div (grad } V_E) = \frac{\partial^2 V_E}{\partial x^2} + \frac{\partial^2 V_E}{\partial y^2} + \frac{\partial^2 V_E}{\partial z^2} = -\frac{\rho}{K}. \quad (106)$$

This is *Poisson's equation*. In general, the charge density ρ is a function of x, y, and z. In a region free of charges, $\rho = 0$, and Eq. (106) reduces to Laplace's equation.

Since curl $\mathbf{H} = 0$, there exists a V_M such that

$$-\mathbf{H} = \text{grad } V_M = \frac{\partial V_M}{\partial x}\,\mathbf{i} + \frac{\partial V_M}{\partial y}\,\mathbf{j} + \frac{\partial V_M}{\partial z}\,\mathbf{k}. \quad (107)$$

V_M is called the *magnetic scalar potential function*. And to make div $\mathbf{H} = 0$, we must have

$$\text{div (grad } V_M) = \frac{\partial^2 V_M}{\partial x^2} + \frac{\partial^2 V_M}{\partial y^2} + \frac{\partial^2 V_M}{\partial z^2} = 0. \quad (108)$$

This is *Laplace's equation*.

If ρ is given, boundary conditions on E or V_E together with Eq. (106) determine V_E, and the electrostatic field E is then found from Eq. (105). Similarly, boundary conditions on H or V_M together with Eq. (108) determine V_M, and the magnetostatic field is then found from Eq. (107). The E and H so found will necessarily satisfy the system of Eqs. (104).

36. Electromagnetic Waves. Radiation. Skin Effect

We may eliminate H from the system of Eqs. (100) by the following procedure. Take the curl of both members of the first equation. This gives

$$\text{curl (curl E}) = \text{curl}\left(-\frac{\partial \mathbf{H}}{\partial t}\right) = -\frac{\partial(\text{curl } \mathbf{H})}{\partial t} \quad (109)$$

Differentiation of the second equation of Eq. (100) leads to

$$\frac{\partial(\text{curl } \mathbf{H})}{\partial t} = \sigma\,\frac{\partial \mathbf{E}}{\partial t} + K\,\frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (110)$$

The left member of Eq. (109) may be transformed by the identity curl (curl E) = grad (div E) − ∇^2E,

$$\text{where } \nabla^2 \mathbf{E} = \frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2}. \quad (111)$$

This holds for any vector function, and could be verified by a lengthy elementary calculation, or more briefly by using symbolic vector operators.   Let us use the fourth equation of Eq. (100) to replace div E by ρ/K, and then substitute the values given by Eqs. (110) and (111) in Eq. (109).   The result is

$$\frac{\text{grad } \rho}{K} - \nabla^2 \mathbf{E} = -\mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu K \frac{\partial^2 \mathbf{E}}{\partial t^2}. \quad (112)$$

By a similar procedure which starts with the second equation, we may eliminate E from the system of Eqs. (100).   The result is

$$\nabla^2 \mathbf{H} = \mu\sigma \frac{\partial \mathbf{H}}{\partial t} + \mu K \frac{\partial^2 \mathbf{H}}{\partial t^2}. \quad (113)$$

Let us consider an electromagnetic field in empty space.   Since there are no electric charges or conduction currents, $\rho = 0$ and $\sigma = 0$.   We use the subscript 0 to indicate that μ and K have the values for empty space, $\mu_0 = 4\pi \times 10^{-7} = 1.257 \times 10^{-6}$ henry per meter and $K_0 = 8.854 \times 10^{-12}$ farad per meter.   Then Eqs. (100) become

$$\text{curl } \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}, \qquad \text{div } \mathbf{E} = 0,$$
$$\text{curl } \mathbf{H} = K_0 \frac{\partial \mathbf{E}}{\partial t}. \qquad \text{div } \mathbf{H} = 0. \quad (114)$$

And we may reduce Eqs. (112) and (113) to the form

$$\nabla^2 \mathbf{E} = \mu_0 K_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \qquad \nabla^2 \mathbf{H} = \mu_0 K_0 \frac{\partial^2 \mathbf{H}}{\partial t^2}. \quad (115)$$

Introduce the constant c defined by the relation

$$c = \frac{1}{\sqrt{\mu_0 K_0}} = 2.998 \times 10^8 \text{ m./sec.} \quad (116)$$

Then Eqs. (115) show that each of the components E_x, E_y, E_z, H_x, H_y, H_z satisfies an equation of the form

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \tag{117}$$

This is similar to Eq. (84) with $v = c$. And the discussion of Eq. (81), a special case of Eq. (84), showed that its solutions included representations of waves traveling with velocity $v = c$. Hence Eq. (117) indicates that electric and magnetic disturbances are propagated through empty space with a common velocity c. And the value of c, given in Eq. (116), is numerically equal to the velocity of light in empty space. These facts are basic in the electromagnetic theory of light.

The conditions which hold in empty space are closely approximated in the atmosphere, provided that there are no electric charges, ions and electrons, present. Hence Eqs. (114) to (117) are useful in studying the fields due to radiation from an antenna. These equations are also applicable to the fields inside hollow wave guides of conducting material.

In a conducting medium, ρ is constant so that grad $\rho = 0$. Hence Eq. (112) reduces to

$$\nabla^2 \mathbf{E} = \mu\sigma\frac{\partial \mathbf{E}}{\partial t} + \mu K \frac{\partial^2 \mathbf{E}}{\partial t^2}. \tag{118}$$

From this and the last equation of Eqs. (99), $J = \sigma E$, we also have

$$\nabla^2 \mathbf{J} = \mu\sigma\frac{\partial \mathbf{J}}{\partial t} + \mu K \frac{\partial^2 \mathbf{J}}{\partial t^2}. \tag{119}$$

A comparison of Eqs. (113), (118), and (119) shows that in a conducting medium **H**, **E**, and **J** satisfy equations of the same form.

For ordinary metal conductors, K is much less than 10^{-11} farad per meter and σ is of the order of 10^7 mhos per meter. Hence if the frequency is less than 10^{10} cycles per second, the first term on the right in Eq. (119) is more than 10^8 times the last term. Thus

we can neglect the last term, and therefore reduce the equation to

$$\nabla^2 \mathbf{J} = \frac{\partial^2 \mathbf{J}}{\partial x^2} + \frac{\partial^2 \mathbf{J}}{\partial y^2} + \frac{\partial^2 \mathbf{J}}{\partial z^2} = \mu\sigma \frac{\partial \mathbf{J}}{\partial t}. \tag{120}$$

This is the equation governing the skin effect in metal conductors.

37. References

The reader desiring to refresh his knowledge of partial differentiation will find the useful properties of partial derivatives reviewed in Chap. II of the author's *Methods of Advanced Calculus*. And in Chaps. VIII and XII of this same book will be found an introduction to vector analysis and the calculus of variations, which includes the application of these subjects to the derivation and transformation of partial differential equations to which we referred in Secs. 32, 34, and 36.

A fuller discussion of the underlying physical theories is contained in such specialized treatises as H. S. Carslaw's *Mathematical Theory of the Conduction of Heat in Solids*, P. M. Morse's *Vibration and Sound*, and J. A. Stratton's *Electromagnetic Theory*.

EXERCISE XIX

1. Write out that component equation of the second equation of Eq. (100) which contains E_z and $\frac{\partial E_z}{\partial t}$.

2. Use Eqs. (86) and (106) to deduce the equation satisfied by the electrostatic potential V_E in cylindrical coordinates:

$$\frac{\partial^2 V_E}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V_E}{\partial \theta^2} + \frac{1}{r}\frac{\partial V_E}{\partial r} + \frac{\partial^2 V_E}{\partial z^2} = -\frac{\rho}{K}.$$

3. Put $V_E = \epsilon^{-az}U(r,\theta,z)$ in the equation of Prob. 2, and assume that $\rho = 0$. Show that the resulting equation has the form of that in Prob. 15 of Exercise XVI, and from that problem deduce that

$$V_E = \epsilon^{-az}J_n(ar)(c_3 \sin n\theta + c_4 \cos n\theta)$$

is a particular solution of the equation of Prob. 2 when $\rho = 0$.

4. In the equation of Prob. 2 assume that $\rho = 0$, and that V_z is independent of z. Deduce from Prob. 14 of Exercise XVI that

$$V_z = (c_1 r^n + c_2 r^{-n})(c_3 \sin n\theta + c_4 \cos n\theta)$$

is a particular solution of the equation of Prob. 2 when $\rho = 0$.

5. In the equation of Prob. 2 assume that V_z is independent of z and θ, so that V_z and ρ are functions of r only. Show that in this case

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV_z}{dr}\right) = -\frac{\rho}{K}.$$

This implies that $dV_z = -\dfrac{dr}{Kr}(\int r\rho\, dr + c_1)$, so that V_z can be found by a second integration.

6. From Prob. 5 deduce that $V_z = c_1 \ln r + c_2$ is a particular solution of the equation of Prob. 2. when $\rho = 0$.

7. If V_z has the form given in Prob. 6, show that

$$E = -\text{grad } V_z = -\frac{c_1}{r}\, \mathbf{u}_r,$$

where $\mathbf{u}_r = \cos\theta\, \mathbf{i} + \sin\theta\, \mathbf{j}$ is a unit vector in the direction of increasing r.

8. Show that when $\mathbf{E}$ and $\mathbf{H}$ are functions of x and t only, so that all y and z derivatives are zero, and we expand in terms of components, as in Eqs. (101) and (103), the Eqs. (114) for an electromagnetic field in empty space take the form

$$0 = \frac{\partial H_x}{\partial t}, \qquad \frac{\partial E_z}{\partial x} = \mu_0\frac{\partial H_y}{\partial t}, \qquad \frac{\partial E_y}{\partial x} = -\mu_0\frac{\partial H_z}{\partial t}, \qquad \frac{\partial E_x}{\partial x} = 0,$$

$$0 = \frac{\partial E_x}{\partial t}, \qquad -\frac{\partial H_z}{\partial x} = K_0\frac{\partial E_y}{\partial t}, \qquad \frac{\partial H_y}{\partial x} = K_0\frac{\partial E_z}{\partial t}, \qquad \frac{\partial H_x}{\partial x} = 0$$

9. Under the conditions of Prob. 8, Eq. (117) reduces to the form $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$. Using Prob. 10 of Exercise XV, show that this has as its general solution $u = f(x - ct) + g(x + ct)$.

10. From Prob. 8 we must have $E_x = $ const. and $H_x = $ const.,

since all their partial derivatives are zero. And from Prob. 9 we must have $E_y = f(x - ct) + g(x + ct)$, and

$$E_z = F(x - ct) + G(x + ct).$$

Show that from these and from Prob. 8 we must have

$$H_y = -K_0 c F(x - ct) + K_0 c G(x + ct) + C_2,$$

and $H_z = K_0 c f(x - ct) - K_0 c g(x + ct) + C_3$, where C_2 and C_3 are constants which may be made zero by redefining the arbitrary functions f, g, F, and G.

11. By direct substitution, and use of the relation (116), show that the values found in Prob. 10 satisfy all the equations of Prob. 8. They give the general solution of the system.

12. By an argument like that used for Eqs. (81) and (82) show that the special case of the solution in Prob. 10 given by $E_x = 0$, $E_y = f(x - ct)$, $E_z = F(x - ct)$, $H_x = 0$, $H_y = -K_0 c F(x - ct)$, $H_z = K_0 c f(x - ct)$ represents a wave moving with velocity c in the direction of the positive x axis.

13. Show that for the advancing transverse plane wave of Prob. 12 the vectors **E** and **H** are perpendicular.

14. Show that $u = U(y,z)[c_1 \cos (\omega t - x) + c_2 \sin (\omega t - x)]$ will be a solution of Eq. (117), provided that

$$\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \left(\frac{\omega^2}{c^2} - \beta^2\right) U = 0.$$

This equation and vectors **E** and **H**, whose components have the special form which gave rise to it, play a central role in the study of hollow wave guides.

15. Show that if we introduce polar coordinates in the yz plane, and call the constant expression in parentheses a^2, the equation of Prob. 14 takes the form for which particular solutions were found in Prob. 15 of Exercise XVI. These particular solutions are used in studying wave guides with circular cross sections.

16. In a long straight wire carrying alternating current the current density **J** satisfies Eq. (120). If the wire is parallel to the z axis, $J_x = 0$, $J_y = 0$, and J_z is independent of z and θ, where

r, θ, z are cylindrical coordinates.   Use Eq. (86) to show this case

$$\frac{\partial^2 J_z}{\partial r^2} + \frac{1}{r}\frac{\partial J_z}{\partial r} = \mu\sigma\frac{\partial J_z}{\partial t}.$$

17. If the current in Prob. 16 has the frequency ω, v write $J_z = u \sin(\omega t + \phi) = \text{Im } U\epsilon^{j\omega t}$.   Here Im means nary component as in Sec. 12, u is a real function of r on $U = u\epsilon^{j\phi}$ is a complex function of r only.   By substituting equation of Prob. 16, derive the ordinary differential equa

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - j\omega\mu\sigma U = 0.$$

It follows from Prob. 15 of Exercise XVI, with $n = 0$ $U = c_1 J_0(ar) + c_2 Y_0(ar)$ will be a solution of this equa $a^2 = -j\omega\mu\sigma$, or $a = j^{3/2}\sqrt{\omega\mu\sigma}$.   If $J_0(x)$ is Bessel's funct order zero defined by the series in Prob. 16 of Exercise XV independent solution $Y_0(x)$ becomes infinite for $x = 0$. leads us to take $c_2 = 0$ in the present case, and to set

$$U = c_1 J_0(j^{3/2}mr),$$

with $m = \sqrt{\omega\mu\sigma}$.   Since $J_0(0) = 1$, at the center of th $r = 0$ and $J_z = \text{Im } c_1\epsilon^{j\omega t}$.   This determines the complex co c_1.   The real functions ber (Bessel real) and bei (Bessel i nary) such that $J_0(j^{3/2}x) = \text{ber } x + j \text{ bei } x$ are tabulated, a as the functions $M(x)$, $\theta(x)$ derived from them and the re ber $x + j$ bei $x = Me^{j\theta}$.   Hence if $J_z = u_0 \sin(\omega t + \beta)$ a center of the wire, the required current density is

$$J_z = Mu_0 \sin(\omega t + \beta + \theta),$$

where M and θ are found from tables of $M(x)$, $\theta(x)$ with x and $m = \sqrt{\omega\mu\sigma}$.

CHAPTER 4

BOUNDARY VALUE PROBLEMS

In Chap. 3 we saw how partial differential equations arose in the study of heat, electricity, mechanical vibrations, and other physical phenomena. For some of the equations, or systems of equations, general solutions containing arbitrary functions could be found. In such cases, for any particular physical problem it must be possible to find auxiliary conditions which determine the proper choice of the arbitrary functions. Such auxiliary relations are often equations which hold for some fixed time, as initial conditions for $t = 0$, or equations which hold for certain fixed points, as boundary conditions for points on the boundary of a segment or region. We shall refer to any set of auxiliary relations as *boundary conditions*. Whether or not the appropriate system of partial differential equations admits of a general solution, in addition to this system a specific physical situation must imply enough auxiliary relations so that there is only one set of values of the unknown variables which satisfy the boundary conditions as well as the differential equations. A system of differential equations combined with a set of boundary conditions which completely determine a solution constitutes a *boundary value problem*.

It is often possible to solve a boundary value problem in the form of a series of terms, each of which satisfies some of the boundary conditions. In the cases discussed, the sum of the series satisfies these some boundary conditions for arbitrary values of certain coefficients. And these coefficients may be determined so that the remaining boundary conditions are satisfied. This last step frequently depends on Fourier expansions like those of Chap. 2. In this chapter we shall solve

a number of boundary value problems by the process just described.

38. Laplace's Equation

It was shown in Eq. (8), Sec. 26, that the temperature in a steady-state distribution satisfies *Laplace's equation*

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0. \tag{1}$$

This equation is also satisfied by the electric potential function in an electrostatic field in space free of charges, Eq. (106) of Sec. 35, with $\rho = 0$, as well as the magnetic potential function in a magnetostatic field, Eq. (107) of Sec. 35. Similarly Eq. (1) holds for gravitational potential in space free of attracting matter. And in irrotational, steady fluid motion the velocity potential is found to satisfy Laplace's equation, Eq. (1). When seeking steady-state or equilibrium solutions whose time derivatives are zero, Eq. (84) of Sec. 31, Eq. (117) of Sec. 36, and the components of Eqs. (113), (118), and (119) of Sec. 36 all reduce to a form equivalent to Eq. (1).

Since the temperatures on the surface of a body determine the steady-state distribution inside, physical considerations suggest that there is just one solution of Eq. (1) taking on given boundary values and it may be proved mathematically that this is the case if the boundary and boundary values are sufficiently regular.

In temperature distribution or potential problems in a plane, taken as the xy plane, Laplace's equation becomes

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \tag{2}$$

Although the general solution of this equation was found in Prob. 9 of Exercise XV, it is not easy to find what values of the arbitrary functions will satisfy given boundary conditions. Hence it is more convenient to use the method of combining particular solutions described in Sec. 30. We illustrate this for

some particular physical situations. For definiteness, we shall use the language of heat flow.

39. Temperatures in a Rectangular Plate

If a homogeneous plane plate has its faces insulated and its edges kept at prescribed temperatures, its steady-state temperatures will be determined. In particular, consider the rectangle $ABCD$ of Fig. 41, with sides AD and BC so long compared with AB and CD that we may treat them as infinite. Let the prescribed boundary temperatures be 0° for AD, DC, and BC and 25° along AB, and assume

$$AB = 20 \text{ cm.}$$

Take the origin at A, and the axes of x and y along AB and AD, respectively. Then the boundary conditions are
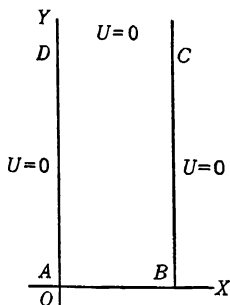
FIG. 41.

$$U(0,y) = 0, \quad U(20,y) = 0, \quad U(x,+\infty) = 0, \quad U(x,0) = 25. \quad (3)$$

Our boundary value problem is to find the solution of Eq. (2) which satisfies the conditions of Eq. (3).

It follows from Eq. (63) of Sec. 30 that Eq. (2) admits

$$U = (c_1 \sin ax + c_2 \cos ax)(c_3 e^{ay} + c_4 e^{-ay}) \quad (4)$$

as a particular solution of the form $X(x) \cdot Y(y)$. By specializing the constants in Eq. (4) we may satisfy the first three conditions of Eq. (3). In fact, if $c_3 = 0$, the second factor will be zero for $y = +\infty$. And if $c_2 = 0$, the first factor will reduce to $c_1 \sin ax$ and thus be zero for $x = 0$. It will also be zero for $x = 20$ if

$$\sin 20a = 0, \quad 20a = n\pi \quad \text{or} \quad a = \frac{n\pi}{20}, \quad (5)$$

where n is any positive integer. If we write B_n for the product

$c_1 c_4$ which goes with a particular n, we have

$$B_n e^{-n\pi y/20} \sin \frac{n\pi x}{20}, \qquad n = 1, 2, 3, \cdots \qquad (6)$$

as a set of terms each of which satisfies Eq. (2) and the first three boundary conditions of Eq. (3).

The same will be true of a sum, or infinite series, of such terms. Hence to solve our boundary value problem it merely remains to determine the coefficients of a series

$$U(x,y) = \sum_{n=1}^{\infty} B_n e^{-n\pi y/20} \sin \frac{n\pi x}{20}, \qquad (7)$$

so that the fourth and last boundary condition of Eq. (3) will be satisfied. Since $e^0 = 1$, this will be the case if

$$25 = U(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{20}, \qquad \text{for } 0 < x < 20. \qquad (8)$$

It follows from the discussion of Fourier sine series based on Eqs. (61) and (62) of Sec. 19 that

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \qquad \text{for } 0 < x < L \qquad (9)$$

for any piecewise regular function $f(x)$, if the b_n are found from

$$b_n = \frac{2}{L} \int_0^L f(x) \sin n\omega x \, dx, \qquad \text{where } \omega = \frac{\pi}{L}. \qquad (10)$$

And from Prob. 24 of Exercise IX we may conclude that in particular

$$Ax + B = \frac{1}{\pi} \left(p \sin \frac{\pi x}{L} - \frac{q}{2} \sin \frac{2\pi x}{L} + \frac{p}{3} \sin \frac{3\pi x}{L} \right.$$
$$\left. - \frac{q}{4} \sin \frac{4\pi x}{L} + \cdots \right), \qquad \text{for } \theta < x < L,$$

$$\text{where } p = 4B + 2LA \text{ and } q = 2LA. \qquad (11)$$

A comparison of Eq. (8) with Eq. (9) shows that the B_n are the

coefficients b_n of the expansion of 25 in a Fourier sine series with frequency $\pi/20 = \pi/L = \omega$ and half-period $L = 20$. The values of b_n could be found from Eq. (10) with $f(x) = 25$. But since $25 = Ax + B$, with $A = 0$, $B = 25$, it is simpler to substitute these values and $L = 20$ in Eq. (11) and so find that $p = 100$, $q = 0$, and

$$25 = \frac{100}{\pi}\left(\sin\frac{\pi x}{20} + \frac{1}{3}\sin\frac{3\pi x}{20} + \frac{1}{5}\sin\frac{5\pi x}{20} + \cdots\right), \quad (12)$$

for $0 < x < 20$.

We may now replace the B_n in Eq. (7) by the values found by identifying Eq. (12) with Eq. (8). An equivalent practical procedure is to multiply each sine term in Eq. (12) by an appropriate exponential factor and thus obtain as the complete solution of our problem

$$U(x,y) = \frac{100}{\pi}\left(e^{-\pi y/20}\sin\frac{\pi x}{20} + \frac{1}{3}e^{-3\pi y/20}\sin\frac{3\pi x}{20}\right.$$
$$\left. + \frac{1}{5}e^{-5\pi y/20}\sin\frac{5\pi x}{20} + \cdots\right). \quad (13)$$

The series may be used for practical computation, provided that y is not too small compared with 20, since the first few terms will then give a good approximation. For example, when $x = 10$, $y = 10$,

$$U(10,10) = \frac{100}{\pi}\left(e^{-\pi/2} - \frac{1}{3}e^{-3\pi/2} + \frac{1}{5}e^{-5\pi/2} - \cdots\right)$$
$$= \frac{100}{\pi}(0.2079 - 0.0030 + 0.0001) = 6.53. \quad (14)$$

Let us next consider the steady-state temperatures for a long rectangle $ABCD$ as determined by a different set of boundary conditions. We again take the prescribed temperatures as $0°$ for AD, DC, and BC. But along AB let the temperature be any given function of the distance from A, $f(x)$. And let us now assume that $AB = L$ cm. Then

$$U(0,y) = 0, \quad U(L,y) = 0, \quad U(x, +\infty) = 0, \quad U(x,0) = f(x). \quad (15)$$

In place of Eq. (7) we now have

$$U(x,y) = \sum_{n=1}^{\infty} B_n e^{-n\pi y/L} \sin \frac{n\pi x}{L}, \qquad (16)$$

which satisfies Eq. (2) and the first three boundary conditions of Eq. (15). It will also satisfy the last boundary condition if we replace the B_n in Eq. (16) by the b_n of Eq. (9). Except for certain simple forms of $f(x)$ it is necessary to determine these b_n from Eq. (10) by calculations like those made in Secs. 18 and 19.

When $f(x) = Ax + B$ for $0 < x < L$, we may use Eq. (11) to find the desired Fourier sine series as we did in forming Eq. (12).

When $f(x)$ is a single sine term whose frequency equals $n\pi/L$ for some integral value of n, the desired Fourier sine series consists of this one term. For example, suppose that in Eq. (15) $L = 14$, and $f(x) = 5 \sin (3\pi x/7)$. Since $3\pi/7 = 6\pi/14 = n\pi/L$ for $n = 6$, $b_6 = 5$, and all the other coefficients b_n, $n \neq 6$, are zero. Hence the final solution in this case is

$$U(x,y) = 5e^{-3\pi y/7} \sin \frac{3\pi x}{7} \qquad (17)$$

A similar procedure may be used if $f(x)$ is given as a finite sum of sine terms, and the frequency of each term is an integral multiple of π/L. Thus suppose that in Eq. (15) $L = 14$ and

$$f(x) = 8 \sin \pi x - 3 \sin \frac{\pi x}{2}. \qquad (18)$$

Since $L = 14$, we may write $\pi = 14\pi/L$, and $\pi/2 = 7\pi/L$. Thus in this case all the b_n are zero except $b_{14} = 8$ and $b_7 = -3$. And it follows directly from Eq. (18) that the final solution is

$$U(x,y) = 8e^{-\pi y} \sin \pi x - 3e^{-\pi y/2} \sin \frac{\pi x}{2}. \qquad (19)$$

EXERCISE XX

A long rectangular plate has its surfaces insulated and the two long sides, as well as one of the short sides, maintained at 0°C.

Find an expression for the steady-state temperature $U(x,y)$ if

1. The other short side, $y = 0$, is kept at $50°$ and is 30 cm. long.

2. $U(x,0) = 4x$, and the short side is 6 cm. long.

3. $U(x,0) = 2x - 4$, and the short side is 4 cm. long.

4. $U(x,0) = 100 \sin \dfrac{\pi x}{40}$, and the short side is 40 cm. long.

5. $U(x,0) = 4 \sin \dfrac{\pi x}{3} - 6 \sin \dfrac{\pi x}{5}$ and the short side is 30 cm. long.

6. $U(x,0) = 10$, $0 < x < 4$; $U(x,0) = 0$, $4 < x < 8$ and the short side is 8 cm. long.

7. $U(x,0) = c$, and the short side is L cm. long.

8. $U(x,0) = cx$, and the short side is L cm. long.

9. $U(x,0) = c(2x - L)$ and the short side is L cm. long.

10. Verify that Eq. (4) is the only form of those in Eqs. (63) of Sec. 30 for which some choice of constants would make $Y(y) = 0$ for $y = +\infty$, and $X(x) = 0$ for two different values of x, as 0 and L.

11. Let $Z = e^{\pi(-y+ix)/L}$ and $S(Z) = \displaystyle\sum_{n=1}^{\infty} B_n Z^n$. Show that Eq. (16) is equivalent to $U(x,y) = \text{Im } S(Z)$, where Im means imaginary component, as in Sec. 12.

12. In the series which defines $S(Z)$ in Prob. 11, replace the B_n by the values which gave the solution of Prob. 8. And use the Maclaurin's series $\ln (1 + Z) = Z - \dfrac{Z^2}{2} + \dfrac{Z^3}{3} - \dfrac{Z^4}{4} + \cdots$ to show that $S(Z) = \dfrac{2cL}{\pi} \ln (1 + Z)$. Then deduce from Prob. 11 that

$$U(x,y) = \frac{2cL}{\pi} \tan^{-1} \left(\frac{e^{-\pi y/L} \sin \dfrac{\pi x}{L}}{1 + e^{-\pi y/L} \cos \dfrac{\pi x}{L}} \right),$$

the solution in closed form of Prob. 8, where $U(x,0) = cx$.

13. Verify directly that the closed form for $U(x,y)$ in Prob. 12 satisfies all the boundary conditions given in Prob. 8.

14. In the series which defines $S(Z)$ in Prob. 11, replace the B_n by the values which gave the solution of Prob. 7. And use the Maclaurin's series $\frac{1}{2} \ln \frac{1+Z}{1-Z} = Z + \frac{Z^3}{3} + \frac{Z^5}{5} + \cdots$ to show that $S(Z) = \frac{2c}{\pi} \ln \frac{1+Z}{1-Z}$. Then deduce from Prob. 11 that

$$U(x,y) = \frac{2c}{\pi} \tan^{-1} \left(\frac{\sin \dfrac{\pi x}{L}}{\sinh \dfrac{\pi y}{L}} \right),$$

the solution in closed form of Prob. 7, where $U(x,0) = c$.

15. Verify directly that the closed form for $U(x,y)$ in Prob. 14 satisfies all the boundary conditions given in Prob. 7.

16. With the notation of Prob. 11, show that we may write the solution

$$U(x,y) = \text{Im } S(Z) = \frac{1}{2i} S[e^{-i\pi(x+iy)/L}] - \frac{1}{2i} S[e^{i\pi(x-iy)/L}].$$

It follows directly from this and Prob. 9 of Exercise XV that Im $S(Z)$, and in particular the closed forms of Probs. 12 and 14 satisfy Eq. (2).

17. A rectangular plate is bounded by the lines $x = 0$, $x = L$, $y = 0$, $y = M$. Its surfaces are insulated, and the temperatures along the four edges are given by $U(0,y) = 0$, $U(L,y) = 0$, $U(x,M) = 0$, $U(x,0) = f(x)$. Find particular solutions of Eq. (2) which satisfy the first three boundary conditions by suitably restricting the constants in Eq. (4). And show that a series of such particular solutions

$$U(x,y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi(y - M)}{L}$$

will satisfy $U(x,0) = f(x)$, and hence solve the boundary value

problem, if

$$B_n = - \frac{b_n}{\sinh \dfrac{n\pi M}{L}}$$

where the b_n are the coefficients of Eq. (9) determined from Eq. (10).

18. If the temperatures along the edges of the plate of Prob. 17 are given by $U(0,y) = G(y)$, $U(L,y) = g(y)$, $U(x,M) = F(x)$, $U(x,0) = f(x)$, show that the steady-state temperature $U(x,y)$ may be found by adding together four solutions of Eq. (2)

$$U(x,y) = U_1(x,y) + U_2(x,y) + U_3(x,y) + U_4(x,y),$$

determined by the respective boundary conditions:

$$
\begin{array}{llll}
U_1(0,y) = G(y), & U_1(L,y) = 0, & U_1(x,M) = 0, & U_1(x,0) = 0; \\
U_2(0,y) = 0, & U_2(L,y) = g(y), & U_2(x,M) = 0, & U_2(x,0) = 0; \\
U_3(0,y) = 0, & U_3(L,y) = 0, & U_3(x,M) = F(x), & U_3(x,0) = 0; \\
U_4(0,y) = 0, & U_4(L,y) = 0, & U_4(x,M) = 0, & U_4(x,0) = f(x).
\end{array}
$$

A method of finding U_4 is outlined in Prob. 17. And with slight modifications this method may be used to find U_1, U_2, and U_3.

19. In Prob. 17 let $U(x,0) = c$, so that $f(x) = c$. Show that

$$U(x,y) = \frac{4c}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \frac{\sin \dfrac{n\pi x}{L} \sinh \dfrac{n\pi(M - y)}{L}}{\sinh \dfrac{n\pi M}{r}},$$

where n odd means that $n = 1, 3, 5, \cdots$.

20. Deduce the solution of Prob. 7 from Prob. 19 by letting M tend to plus infinity.

40. Temperatures in a Circular Plate

We shall next find the steady-state temperature distribution of a circular plate whose faces are insulated and whose circumference

is kept at prescribed temperatures. In particular, let one diameter of the circle be AB, and let the radius $OA = 25$ cm. Let the

FIG. 42.

temperature be $0°$ at A and $150°$ at B, and increase linearly along the circumference between those points. Thus in Fig. 42 at C on the upper semicircle $U = 150\theta/\pi$, and at C' on the lower semicircle
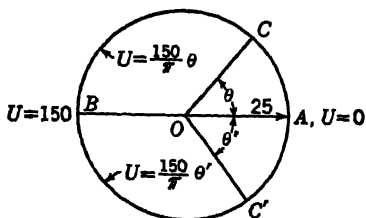
$$U = 150\theta'/\pi.$$

If we use polar coordinates, $\theta' = -\theta$, and the boundary conditions are

$$U(25,\theta) = \frac{150\theta}{\pi}, \quad \text{if } 0 < \theta < \pi,$$

$$= -\frac{150\theta}{\pi}, \quad \text{if } -\pi < \theta < 0. \qquad (20)$$

Since U must satisfy Laplace's equation, $U(r,\theta)$ is a solution of

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{1}{r}\frac{\partial U}{\partial r} = 0, \qquad (21)$$

which is Eq. (2) in polar coordinates by Eq. (86) of Sec. 32.

From Prob. 14 of Exercise XVI, it follows that Eq. (21) admits

$$U = (c_1 r^k + c_2 r^{-k})(c_3 \sin k\theta + c_4 \cos k\theta) \qquad (22)$$

as a particular solution of special form.

From the nature of polar coordinates, $U(r,\theta)$ will be periodic of period 2π. Hence we make the particular solutions, Eq. (22), of period 2π by taking $k = n$ where n is zero or a positive integer. And since r^{-n} is infinite for $r = 0$, we take $c_2 = 0$. Also we write A_n for the product $c_1 c_4$ and B_n for the product $c_1 c_3$ which goes with a particular n. Then

$$A_0 \text{ and } A_n r^n \cos n\theta + B_n r^n \sin n\theta, \qquad n = 1,2,3, \cdots \quad (23)$$

is a set of particular solutions of period 2π, and we write

$$U(r,\theta) = A_0 + \sum_{n=1}^{\infty} (A_n r^n \cos n\theta + B_n r^n \sin n\theta). \quad (24)$$

This will satisfy the boundary condition, Eq. (20), if

$$U(25,\theta) = A_0 + \sum_{n+1}^{\infty} (A_n 25^n \cos n\theta + B_n 25^n \sin n\theta). \quad (25)$$

It follows from the discussion of Fourier series based on Eq. (47) of Sec. 18 and Eq. (55) of Sec. 19 that

$$f(x) = a + \sum_{n=1}^{\infty} (a_n \cos n\omega x + b_n \sin n\omega x), \quad \text{for } -L < x < L \quad (26)$$

for any piecewise regular function $f(x)$, if

$$a = \frac{1}{2L} \int_{-L}^{L} f(x)dx, \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos n\omega x \, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin n\omega x \, dx, \qquad \omega = \frac{\pi}{L} \quad \text{or} \quad L = \frac{\pi}{\omega}. \quad (27)$$

A comparison of Eq. (25) with Eq. (26) shows that $A_0 = a$, $A_n 25^n = a_n$, $B_n 25^n = b_n$, the coefficients in the expansion of $U(25,\theta)$ in a Fourier series with frequency $1 = \omega$ and half-period $L = \pi$. The values of a, a_n, b_n could be found from Eqs. (27) and (20) with $f(x) = U(25,x)$.

Since $U(25,\theta)$ is an even function, the b_n and hence the B_n will all be zero. Also a and the a_n may be found from Eq. (56) of Sec. 19. In fact, since the graph of $U(25,\theta)$ against θ differs from Fig. 32 only by a change in the vertical scale, the coefficients may be obtained from those of Eq. (60) of Sec. 19 by multiplication by $150/\pi$. Hence

$$A_0 = a = 75, \qquad A_n = 25^{-n}a_n = -\frac{600}{\pi^2 n^2} 25^{-n}, \quad \text{if } n \text{ is odd}. \quad (28)$$

Thus the solution of our problem in series form is

$$U(r,\theta) = 75 - \frac{600}{\pi^2}\left(\frac{r}{25}\cos\theta + \frac{r^3}{3^2 \cdot 25^3}\cos 3\theta + \frac{r^5}{5^2 \cdot 25^5}\cos 5\theta + \cdots\right). \quad (29)$$

Let us next consider the steady-state temperature for a circular plate as determined by a different boundary condition. We now assume that the radius of the plate is R cm. And on the circumference let the prescribed temperature be any given function of the polar angle θ, so that $U(R,\theta) = f(\theta)$. Then Eq. (24) still applies, but in place of Eq. (25) we now have

$$f(\theta) = A_0 + \sum_{n=1}^{\infty}(A_n R^n \cos n\theta + B_n R^n \sin n\theta). \quad (30)$$

Comparing with Eq. (26), with $x = \theta$, $\omega = 1$, $L = \pi$, we find that

$$A_0 = a, \qquad A_n = R^{-n}a_n, \qquad B_n = R^{-n}b_n. \quad (31)$$

The Fourier coefficients a, a_n, b_n must be determined from Eq. (27) with $\omega = 1$, $L = \pi$, unless $f(\theta)$ has one of the special properties mentioned in Sec. 19 which permit the use of the simpler formulas there derived. Once a, a_n, b_n are known, we may find A_0, A_n, B_n from Eq. (31). Substitution of these values in Eq. (24) then leads to the solution of the problem.

EXERCISE XXI

Find $U(r,\theta)$, the steady-state temperature distribution of a circular plate of radius R whose surfaces are insulated, if

1. $R = 9$ and $U(9,\theta) = 1$, if $0 < \theta < \pi$, and $= 0$, if $-\pi < \theta < 0$.
2. $R = 20$ and $U(20,\theta) = 80 \sin\theta + 800 \sin 2\theta$.
3. $R = 1$ and $U(1,\theta) = 2\theta$, $\quad 0 < \theta < 2\pi$.
4. $R = 10$ and $U(10,\theta) = 3{,}000\cos(3\theta - 25°)$.

5. A plate in the form of a circular sector is bounded by the lines $\theta = 0$, $\theta = \alpha$, $r = R$. Its surfaces are insulated, and the temperatures along the boundary are $U(r,0) = 0$, $U(r,\alpha) = 0$,

$U(R,\theta) = f(\theta)$. By restricting the constants in Eq. (22), deduce the particular solution that satisfies the first two conditions, $C_n r^{\pi n/\alpha} \sin \dfrac{\pi n \theta}{\alpha}$.

6. Using a series of the particular solutions of Prob. 5, find $U(r,\theta)$ if $\alpha = \pi/3$ and $f(\theta) = 100$.

7. A plate in the form of a ring is bounded by the lines $r = 2$, $r = 4$. Its surfaces are insulated, and the temperatures along the boundary are $U(2,\theta) = 10 \sin \theta + 6 \cos \theta$,

$$U(4,\theta) = 17 \sin \theta + 15 \cos \theta.$$

Using a series of particular solutions of the appropriate form here $(C_n r^n + D_n r^{-n}) \cos n\theta + (E_n r^n + F_n r^{-n}) \sin n\theta$, find the steady-state temperature in the ring, $U(r,\theta)$.

8. Let $U(r,\theta)$ be the steady-state temperature distribution of a circular plate of radius R whose surfaces are insulated. If $U(R,\theta) = f(\theta)$, show that the solution may be written as

$$U(r,\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{R} \right)^n \cos n(\theta - t) \right] f(t)\,dt.$$

9. If Re means real component as in Sec. 12, show that $\left(\dfrac{r}{R} \right)^n \cos n(\theta - t) = \text{Re}\left(\dfrac{z}{Z} \right)^n$, if $z = re^{i\theta}$, $Z = Re^{it}$. Hence the bracket in Prob. 8 is the real component of the geometric series $\dfrac{1}{2} + \sum_{n=1}^{\infty} \left(\dfrac{z}{Z} \right)^n = \dfrac{Z + z}{Z - z}$. Use this fact to reduce the solution of Prob. 8 to the form

$$U(r,\theta) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr \cos (t - \theta) + r^2} f(t)\,dt.$$

This expression for the solution of Laplace's equation, Eq. (21), in terms of prescribed values on the boundary of a circular region is known as *Poisson's integral*.

10. Use Prob. 9 to show that if $U(R,\theta) = c$ for $0 < \theta < \pi$ and

$= 0$ for $-\pi < \theta < 0$, the solution in closed form is

$$U(r,\theta) = \frac{c}{2} + \frac{c}{\pi} \tan^{-1} \left[\frac{2rR \sin \theta}{R^2 - r^2} \right].$$

11. Use Prob. 9 to show that if $U(R,\theta) = c$ for $0 < \theta < \pi$ and $= -c$ for $-\pi < \theta < 0$, the solution in closed form is

$$U(r,\theta) = \frac{2c}{\pi} \tan^{-1} \left[\frac{2rR \sin \theta}{R^2 - r^2} \right]$$

12. Use Prob. 10 to solve Prob. 1 in closed form, and show that this is equivalent to the series found in Prob. 1. HINT: Take imaginary components of the Maclaurin's series in $z = re^{i\theta}$,

$$\frac{1}{2} \ln \frac{R + z}{R - z} = \frac{z}{R} + \frac{z^3}{3R^3} + \frac{z^5}{5R^5} + \cdots$$

to deduce that

$$\tan^{-1} \left[\frac{2rR \sin \theta}{R^2 - r^2} \right] = \frac{r}{R} \sin \theta + \frac{r^3}{3R^3} \sin 3\theta + \frac{r^5}{5R^5} \sin 5\theta \cdots$$

41. Cooling of a Rod

Consider the changing temperatures of a thin uniform rod, AB in Fig. 43, whose sides are insulated. Let $U(x,t)$ denote the
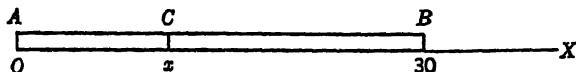
FIG. 43.

temperature of the cross section C for which $AC = x$ at time t. It was shown in Eq. (5) of Sec. 26, that

$$\frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2}. \tag{32}$$

In particular let AB be 30 cm. long, of material for which the thermal diffusivity $a^2 = 2$ cm.^{2}/sec. Suppose that originally all points of the rod were at 15° but that at time $t = 0$ the ends of the rod A and B had their temperatures suddenly changed to 0°

and kept at this temperature. We wish to find $U(x,t)$ for any later time $t > 0$. The initial and boundary conditions are

$$U(x,0) = 15, \qquad U(0,t) = 0, \qquad U(30,t) = 0. \qquad (33)$$

Thus our boundary value problem is to find the solution of Eq. (32) which also satisfies the conditions (33).

It follows from Prob. 1 of Exercise XVI that Eq. (32) admits

$$U = (c_3 \sin bx + c_4 \cos bx)e^{-a^2b^2t} \qquad (34)$$

as a particular solution of the form $X(x) \cdot T(t)$. The second condition of Eq. (33) will be satisfied if $c_4 = 0$. And the third condition will be satisfied if

$$\sin 30b = 0, \quad 30b = n\pi, \quad \text{or} \quad b = \frac{n\pi}{30} \quad \text{and} \quad a^2b^2 = \frac{n^2\pi^2}{450}, \qquad (35)$$

where n is any positive integer. We write B_n for the constant c_3 which goes with a particular n, and have

$$B_n \sin \frac{n\pi x}{30} e^{-n^2\pi^2t/450}, \qquad n = 1,2,3, \cdots \qquad (36)$$

as a set of terms each of which satisfies Eq. (32) and the last two conditions of Eq. (33). Hence we put

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{-n^2\pi^2t/450}. \qquad (37)$$

The remaining initial condition of Eq. (33) will be satisfied if

$$15 = U(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}, \qquad \text{for } 0 < x < 30. \qquad (38)$$

A comparison of this with Eq. (9) shows that the B_n are the coefficients b_n of the expansion of 15 in a Fourier sine series with frequency $\pi/30 = \pi/L = \omega$ and half-period $L = 30$. The values of b_n could be found from Eq. (10) with $f(x) = 15$. But it is simpler to put $A = 0$, $B = 15$, $L = 30$ in Eq. (11) and so find

that $p = 60$, $q = 0$, and

$$15 = \frac{60}{\pi} \left(\sin \frac{\pi x}{30} + \frac{1}{3} \sin \frac{3\pi x}{30} + \frac{1}{5} \sin \frac{5\pi x}{30} + \cdots \right),$$
$$\text{for } 0 < x < 30. \quad (39)$$

We must now replace the B_n in Eq. (37) by the values obtained when we identify Eq. (39) with Eq. (38). The result is

$$U(x,t) = \frac{60}{\pi} \left(\sin \frac{\pi x}{30} e^{-\pi^2 t/450} + \frac{1}{3} \sin \frac{3\pi x}{30} e^{-\pi^2 t/50} + \cdots \right). \quad (40)$$

For large values of t this series converges very rapidly owing to the exponential terms. In fact, if t exceeds 20, the first two terms give a result correct to within one part in ten thousand.

The process used above requires modification if the ends of the rod are suddenly changed to fixed temperatures different from zero. We illustrate the revised procedure by solving the following problem.

Suppose that the rod of Fig. 43 has end A kept at 30° and end B kept at 120° until temperatures indistinguishable from the steady state are reached. We again assume $AB = 30$ cm. and $a^2 = 2$ cm.^{2}/sec. At some later time, let us suddenly lower the temperature of A to 15° and that of B to 45°, and from then on maintain these temperatures. We wish to find $U(x,t)$ for $t > 0$, where t is measured from the sudden change.

We must make use of the steady-state solution of Eq. (32). Since this is independent of t, it will be a solution of

$$\frac{d^2 U}{dx^2} = 0, \quad \text{or} \quad U = c_1 x + c_2. \quad (41)$$

We first find constants c_1 and c_2 such that this takes on the original end values, 30 at A, $x = 0$, and 120 at B, $x = 30$. Hence

$$30 = c_2, \ 120 = 30c_1 + c_2 \quad \text{and} \quad c_2 = 30, \ c_1 = 3. \quad (42)$$

This shows that $3x + 30$ was the temperature before the sudden change and, for our problem, the initial condition is

$$U(x,0) = 3x + 30. \quad (43)$$

The temperatures for our problem will approach the steady-state solution for the changed end values, 15 at A, $x = 0$, and 45 at B, $x = 30$. The values of c_1 and c_2 which make the U of Eq. (41) take on these values are found from

$$15 = c_2, 45 = 30c_1 + c_2 \qquad \text{to be} \qquad c_2 = 15, c_1 = 1. \quad (44)$$

This shows that the steady-state solution

$$U_S = x + 15 \tag{45}$$

satisfies the changed boundary conditions for our problem

$$U(0,t) = 15, \qquad U(30,t) = 45. \tag{46}$$

Mathematically considered, our problem is to find a solution of Eq. (32) that satisfies the initial condition (43) and the boundary conditions (46). We transform this to a problem with end values zero by putting

$$U = U_s + U_T = x + 15 + U_T. \tag{47}$$

This makes the new function U_T satisfy the relation

$$U_T = U - U_s = U - x - 15. \tag{48}$$

Since U and U_s are each solutions of Eq. (32), their difference U_T is also a solution. It follows from the way in which U_s was determined, and may be verified from Eqs. (48) and (46) that

$$U_T(0,t) = 0, \qquad U_T(30,t) = 0. \tag{49}$$

Also from Eqs. (48) and (43) we may deduce that

$$U_T(x,0) = U(x,0) - x - 15 = 2x + 15. \tag{50}$$

The problem of finding $U_T(x,t)$, a solution of Eq. (32) which satisfies the initial condition (50) and the boundary conditions (49), is similar in type to the first problem solved in this section. To solve it, we begin with the relation similar to Eq. (37),

$$U_T(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30} e^{-n^2\pi^2 t/450}. \tag{51}$$

This satisfies the differential equation (32) and the end conditions
(49). The initial condition (50) will also be satisfied if

$$2x + 15 = U_T(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{30}.$$ (52)

The coefficients of this Fourier sine series of frequency $\pi/30$ and
half-period $L = 30$ may be found by putting $A = 2$, $B = 15$,
$L = 30$ in Eq. (11).   This makes $p = 180$, $q = 120$, so that

$$B_n = \frac{180}{\pi n} \qquad \text{for } odd \ n = 1,3,5, \cdot \cdot \cdot$$

and

$$B_n = \frac{120}{\pi n} \qquad \text{for } even \ n = 2,4,6, \cdot \cdot \cdot \cdot .$$ (53)

On substituting these values in Eq. (51) we obtain the transient
part of the solution, U_T. And from this and Eq. (47) we find
that

$$U(x,t) = x + 15 + \frac{1}{\pi} \left(180 \sin \frac{\pi x}{30} e^{-\pi^2 t/450} - 60 \sin \frac{2\pi x}{30} e^{-2\pi^2 t/225} \right.$$
$$\left. + 60 \sin \frac{3\pi x}{30} e^{-\pi^2 t/50} - \cdot \cdot \cdot \right).$$ (54)

Let us outline our methods of solution in more general terms.
First suppose that originally the temperature was some given
function of the distance, $g(x)$, and that at time $t = 0$ the ends of
the rod A and B had their temperatures suddenly changed to $0°$
and kept at this temperature. We now assume that $AB = L$ cm.
and write a^2 for the thermal diffusivity. The boundary condi-
tions now are

$$U(x,0) = g(x), \qquad U(0,t) = 0, \qquad U(L,t) = 0.$$ (55)

And in place of Eq. (37) in this case we have

$$U(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-a^2 n^2 \pi^2 t/L^2},$$ (56)

which satisfies Eq. (32) and the last two boundary conditions of
Eq. (55). It will also satisfy the last boundary condition if we
replace the B_n in Eq. (56) by the b_n of Eq. (9) with $f(x) = g(x)$.
We determine these b_n from Eq. (10) or one of the alternatives to
it mentioned in Sec. 39.

Next consider the same rod of length L with initial temperature
$g(x)$, but assume that at time $t = 0$ the end A had its tem-
perature changed to $r°$ and that the end B had its temperature $s°$,
and both ends kept at the new temperatures. Here the boundary
conditions are

$$U(x,0) = g(x), \qquad U(0,t) = r, \qquad U(L,t) = s. \qquad (57)$$

We transform this to a problem involving U_T by writing

$$U = U_s + U_T = \frac{s - r}{L} x + r + U_T, \qquad (58)$$

in which we have replaced U_s by an expression like that in Eq.
(41) with constants such that the end conditions of Eq. (57) are
satisfied. The boundary conditions for U_T in this case are

$$U_T(x,0) = g(x) - \frac{s - r}{L} x - r, \; U_T(0,y) = 0, \; U_T(L,t) = 0. \quad (59)$$

The determination of $U_T(x,t)$ from these conditions and Eq. (32)
is similar in type to the problem of finding $U(x,t)$ from Eqs. (55)
and (32). Thus we may find U_T from a series like that in Eq.
(56),

$$U_T(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-a^2 n^2 \pi^2 t / L^2}. \qquad (60)$$

This will satisfy the first condition of Eq. (59) if we replace the
B_n in Eq. (60) by the b_n of Eq. (9) with $f(x) = g(x) - \frac{s - r}{L} x - r$.
We determine these b_n from Eq. (10) or one of the alternatives to
it mentioned in Sec. 39. We may then insert the values of B_n in

$$U(x,t) = \frac{s - r}{L} x + r + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} e^{-a^2 n^2 \pi^2 t / L^2}, \qquad (61)$$

obtained from Eqs. (58) and (60), to give the solution of the problem with boundary conditions (57).

EXERCISE XXII

The ends A and B of a rod 50 cm. long have their temperatures kept at 0° and 100°, respectively, until the temperatures are indistinguishable from those for the steady state. At some time after this, there is a sudden change. Find the temperature of any point in the rod, $U(x,t)$, as a function of x cm., the distance from A, and t sec., the time elapsed after the sudden change, if the new temperatures maintained at A and B are

1. 0° at A, 0° at B.   **2.** 100° at A, 100° at B.
3. 0° at A, 50° at B.   **4.** 50° at A, 0° at B.
5. 25° at A, 75° at B.   **6.** 50° at A, 150° at B.

7. In Prob. 1 compute the temperature at the mid-point 2 min. after the sudden change if the rod is made of silver for which $a^2 = 1.74$.

8. A rod 80 cm. long has one half its length at 0°, and the other half at 50°. If the sides are suddenly insulated, the temperature of the hot end reduced to 0°, and from then on the two ends are kept at 0°, find the temperature of the rod $U(x,t)$ as a function of the distance from the end originally at 0° and the time after the sudden change.

9. In Prob. 8 compute the temperature for a point 20 cm. from the end originally at 0°, 1 hr. after the sudden change if the rod is made of wrought iron for which $a^2 = 0.173$.

If the rod is made of glass for which $a^2 = 0.00571$, compute the temperature at a point 5 cm. from A, 3 hr. after the sudden change

10. For the rod of Prob. 2.   **11.** For the rod of Prob. 6.

In Prob. 5 compute the temperature at a point 12.5 cm. from A, 5 min. after the sudden change if

12. The rod is made of silver for which $a^2 = 1.74$.
13. The rod is made of wrought iron for which $a^2 = 0.173$.
14. The rod is made of glass for which $a^2 = 0.00571$.

15. Evaluate the result of Prob. 5 for $x = 25$. Note that

$U(25,t)$ does not depend on t or a^2 in this case.   This shows that at the mid-point of the rod of Prob. 5 the temperature does not change with the time and is also independent of the kind of homogeneous material out of which the rod is made.

16. The ends of a rod 60 cm. long are insulated so that $\dfrac{\partial U}{\partial x} = 0$ at $x = 0$ and $\dfrac{\partial U}{\partial x} = 0$ at $x = 60$.   Show that for $n = 0, 1, 2, 3, \cdots, A_n \cos \dfrac{n\pi x}{60} e^{-a^2 n^2 \pi^2 t/3,600}$ is a particular solution of Eq. (32) satisfying the end conditions.

17. Suppose that the rod of Prob. 16 had its end points kept at $0°$ and $180°$ until the steady state was approximated and that the ends were then suddenly insulated at $t = 0$.   Derive the proper initial condition $U(x,0) = 3x$ and, using a series of particular solutions of the type found in Prob. 16, find $U(x,t)$. Take $a^2 = 2$.

18. A rod AB is 60 cm. long, of material for which $a^2 = 2$. Originally all of its points were at $0°$.   From $t = 0$ on, heat was supplied to the rod through the end A at a constant rate of $120KA$ cal./sec.   By Sec. 26, $-KA \dfrac{\partial U}{\partial x} = 120KA$, so that $\dfrac{\partial U}{\partial x} = -120$ at $x = 0$ for $t > 0$.   And at $t = 0$ the end B was suddenly insulated, so that $\dfrac{\partial U}{\partial x} = 0$ at $x = 60$ for $t > 0$.   Show that $A(x^2 + 2a^2 t) + Bx$ is a particular solution of Eq. (32), and that it will satisfy the end conditions if $A = 1$, $B = -120$.   Now put $U(x,t) = x^2 - 120x + 4t + U_1(x,t)$ and find $U_1(x,t)$ by the method of Prob. 17.

19. A rod AB is 20 cm. long.   The end A is kept at $0°$ so that $U(0,t) = 0$.   The end B is insulated so that $\dfrac{\partial U}{\partial x} = 0$ at $x = 20$. Show that for $n = 1, 3, 5, \cdots$, any odd integer,

$$B_n \sin \frac{n\pi x}{40} e^{-a^2 n^2 \pi^2 t/1,600}$$

is a particular solution of Eq. (32) satisfying the end conditions.

20. Suppose that just before the end conditions of Prob. 19

were imposed the temperature distribution was $U(x,0) = 5x$. Using a series of particular solutions of the type found in Prob. 19, find $U(x,t)$. The coefficients in the expansion of $5x$ in an odd-harmonic sine series of period 80, valid for $0 < x < 20$ can be found from integrals taken over this interval. This follows from an argument like that at the end of Sec. 19, based on consideration of an odd, odd-harmonic function of period 80 equal to $5x$ for $0 < x < 20$.

21. A rod 40 cm. long has its mid-point raised to 100° while the ends are at 0° until each half approximates the steady state. The source of heat is then removed from the mid-point and this is insulated. Find the temperature $U(x,t)$.

22. From physical considerations, deduce that the solution of Prob. 21 is symmetrical about the mid-point and gives the solution of Prob. 20 when x is restricted to values between 0 and 20.

23. The equation for the flow of heat in a rod with radiating surfaces is $\dfrac{\partial U}{\partial t} = a^2 \dfrac{\partial^2 U}{\partial x^2} - b^2(U - U_0)$. Show that if $U_0 = 0$, for any value of k, $e^{-(a^2k^2+b^2)t} (A \cos kx + B \sin kx)$ is a particular solution.

24. Find the temperature $U(x,t)$ of the radiating rod of Prob. 23 if $U_0 = 0$, $U(0,t) = 0$, $U(10,t) = 0$, $U(x,0) = x$. HINT: Use a sum of particular solutions with $A = 0$, $k = n\pi/10$. Take $a^2 = 2$, $b^2 = 5$.

25. Show that $U = U_0 + e^{-b^2t}V$ will be a solution of the equation of Prob. 23 if $V(x,t)$ is a solution of Eq. (32). Use this to check Prob. 24.

26. The *Fourier integral* of Sec. 22 may be used to combine particular solutions of Eq. (32). With $b = u$, $c_3 = \cos uv$, $c_4 = \sin uv$ the particular solution in Eq. (34) becomes

$$\cos u(x - v)e^{-a^2u^2t}.$$

Equation (111) of Sec. 22 suggests that we form the expression

$$U(x,t) = \frac{1}{\pi} \int_0^\infty du \int_{-\infty}^\infty f(v)e^{-a^2u^2t} \cos u(x - v)dv.$$

If $f(x)$ is such that differentiation inside the double infinite integral is legitimate, show that this will be a solution of Eq. (32). Also show that if $t = 0$, the expression is equivalent to the right member of Eq. (111) of Sec. 22, and thus reduces to $f(x)$.

Thus for suitable $f(x)$ defined for $-\infty < x < \infty$, the equation just written gives the solution of the boundary value problem made up of Eq. (32) and $U(x,0) = f(x)$ for $-\infty < x < \infty$.

27. By using contour integration in the complex plane, and the fact that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, it may be shown that the definite integral $\int_0^\infty e^{-k^2u^2}\cos bu\,du = \frac{1}{2a}\sqrt{\pi}e^{-b^2/4k^2}$. Invert the order of integration in Prob. 26, and use the result just stated to evaluate the inner integral. By introducing a new variable $w = \dfrac{v - x}{2a\sqrt{t}}$, reduce the result to the form

$$U(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2aw\sqrt{t})e^{-w^2}\,dw.$$

28. By differentiating under the integral sign, and using integration by parts, show that the expression in Prob. 27 satisfies Eq. (32). Also show that it makes $U(x,0) = f(x)$ for

$$-\infty < x < \infty.$$

The operations used are legitimate for some $f(x)$, like that of Prob. 29, for which the Fourier integral used in Prob. 26 diverges.

29. An infinite rod has its initial temperature $U(x,0) = f(x)$, where $f(x) = r$ if $x < 0$ and $f(x) = s$ if $x > 0$. Use Probs. 28 and 27 to find $U(x,t)$ for $t > 0$. And show that

$$U(x,t) = \frac{r}{\sqrt{\pi}} \int_{-\infty}^{-\frac{x}{2a\sqrt{t}}} e^{-w^2}\,dw + \frac{s}{\sqrt{\pi}} \int_{-\frac{x}{2a\sqrt{}}}^{\infty} e^{-w^2}\,dw \text{ for } t > 0.$$

30. The *error function* is defined by erf $x = \dfrac{2}{\sqrt{\pi}} \int_0^x e^{-w^2}\,dw$. As stated in Prob. 27, erf $\infty = 1$. Show that the result of Prob.

29 may be written in the form

$$U(x,t) = \frac{r + s}{2} + \frac{(s - r)}{2} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) \qquad \text{if } x > 0$$

and

$$U(x,t) = \frac{r + s}{2} + \frac{(r - s)}{2} \operatorname{erf}\left(\frac{x}{2a\sqrt{t}}\right) \qquad \text{if } x < 0.$$

31. A semiinfinite rod has its initial temperature $U(x,0) = c$, and $U(0,t) = 0$ for $t > 0$. Find $U(x,t)$ for $x > 0$, $t > 0$. HINT: Put $s = c$, $r = -c$ in Prob. 30 to get $U(x,t) = c \operatorname{erf}\left(\dfrac{x}{2a\sqrt{t}}\right)$.

32. A very long rod has $a^2 = 0.04$. Assume that the rod is on the x axis, and that at $t = 0$ the temperature was 20° for $x < 0$ and 100° for $x > 0$. Use Prob. 30 and tables of erf x to compute U at a point 1.2 m. to the right of the origin after 100 hr., or $U(120, 360,000)$.

42. The Long Transmission Line

In Eq. (96) of Sec. 33 we showed that the equations governing the propagation of potential $e(x,t)$ and current $i(x,t)$ along an
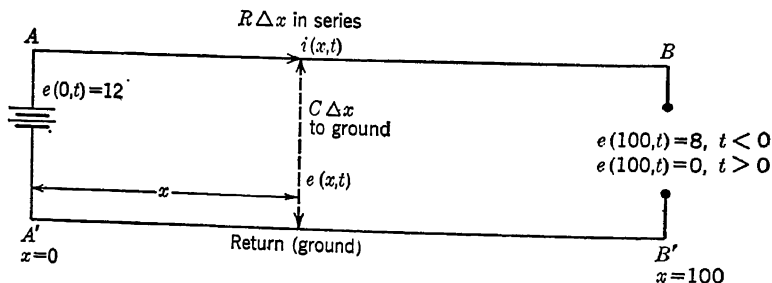
FIG. 44.

electrical cable having a series resistance of R ohms per mile and a shunt capacitance of C farads per mile are the *telegraph equations*

$$-\frac{\partial e}{\partial x} = Ri, \qquad -\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t}, \qquad \frac{\partial^2 e}{\partial x^2} = RC\frac{\partial e}{\partial t}. \qquad (62)$$

The third relation has the same form as Eq. (32). When boundary conditions for e are known, this equation may be solved by the methods given in Sec. 41. And after e has been found, we may obtain the value of i from the first relation of Eq. (62).

As an example, let us consider a line AB 100 miles long for which $R = 0.1$ ohms per mile and $C = 2$ microfarads per mile. Then $C = 2 \times 10^{-6}$ farad per mile, and in Eq. (62) the coefficient $RC = 2 \times 10^{-7}$. Suppose that originally the line was under steady-state conditions, with potential 12 volts at A, $x = 0$, and 8 volts at B, $x = 100$. At some instant, taken as $t = 0$ in Fig. 44, the terminal B is suddenly grounded, reducing its potential to 0 volts. But the potential at A is kept at 12 volts. We wish to find the potential $e(x,t)$ and the current $i(x,t)$ at any point of the line $0 < x < 100$, and at any time subsequent to the grounding, $t > 0$.

The steady-state solution of

$$\frac{\partial^2 e}{\partial x^2} = RC \frac{\partial e}{\partial t}, \qquad \text{or} \qquad \frac{\partial e}{\partial t} = \frac{1}{RC} \frac{\partial^2 e}{\partial x^2}, \qquad (63)$$

will be independent of t and hence a solution of

$$\frac{d^2 e}{dx^2} = 0, \qquad \text{or} \qquad e = c_1 x + c_2. \qquad (64)$$

This will take on the original end values if

$$12 = c_2, 8 = 100c_1 + c_2 \qquad \text{and} \qquad c_2 = 12, c_1 = -0.04. \qquad (65)$$

This shows that the emf just before the sudden change was

$$e(x,0) = -0.04x + 12. \qquad (66)$$

After the sudden change, as t increases, the emf will approach the steady-state solution for the changed end values

$$e(0,t) = 12, \qquad e(100,t) = 0. \qquad (67)$$

The e of Eq. (64) will take on these values if

$$12 = c_2, 0 = 100c_1 + c_2 \qquad \text{and} \qquad c_2 = 12, c_1 = -0.12. \qquad (68)$$

This shows that the steady-state solution of Eq. (63)

$$e_S = -0.12x + 12 \tag{69}$$

satisfies the boundary conditions on e for our problem, Eq. (67).

For the potential $e(x,t)$, our boundary value problem is to find a solution of Eq. (63) that satisfies the initial condition Eq. (66) and the boundary conditions, Eqs. (67). We transform this to a problem with end values zero as we did in Eq. (47). Here we set

$$e = e_S + e_T = -0.12x + 12 + e_T. \tag{70}$$

This makes the transient effect e_T satisfy the relation

$$e_T = e + 0.12x - 12. \tag{71}$$

As the difference of two solutions of Eq. (63), e_T is also a solution. From the way we found e_S, or from Eqs. (71) and (67), we see that

$$e_T(0,t) = 0, \qquad e_T(100,t) = 0. \tag{72}$$

And from Eqs. (71) and (66) we may deduce the initial condition

$$e_T(x,0) = e(x,0) + 0.12x - 12 = 0.08x \tag{73}$$

To obtain a series for e_T, we set $a^2 = 1/RC$ and $L = 100$ in the right member of Eq. (60). We also write ϵ for the exponential base to avoid confusion with e for emf. This leads to

$$e_T(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100} \epsilon^{-n^2\pi^2 t/10^4 RC}, \tag{74}$$

which satisfies Eq. (63) and the boundary conditions (72). It will also satisfy the remaining initial condition (73) if

$$0.08x = e_T(x,0) + \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{100}, \qquad \text{for } 0 < x < 100. \tag{75}$$

This requires the B_n to be the coefficients b_n of the expansion of $0.08x$ in a Fourier sine series with frequency $\pi/100 = \pi/L$ and half-period $L = 100$. The values of b_n could be found from Eq. (10) with $f(x) = 0.08x$. But it is simpler to put $A = 0.08$, $B = 0$,

$L = 100$ in Eq. (11) and thus find that $p = 16$, $q = 16$, and

$$B_n = b_n = (-1)^{n+1} \frac{16}{\pi n}. \tag{76}$$

Now put these values in Eq. (74), with $RC = 2 \times 10^{-7}$, and substitute the series in Eq. (70). This gives the solution for e,

$$e(x,t) = -0.12x + 12 - \frac{16}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \sin \frac{n\pi x}{100} \epsilon^{-500n^2\pi^2t}. \tag{77}$$

The current i may be found from the first relation of Eq. (62), written in the form

$$i = -\frac{1}{R} \frac{\partial e}{\partial x}. \tag{78}$$

We find $\frac{\partial e}{\partial x}$ from Eq. (77), and put the result in Eq. (78) with $R = 0.1$. This gives the solution for i,

$$i(x,t) = 1.2 + 1.6 \sum_{n=1}^{\infty} (-1)^n \cos \frac{n\pi x}{100} \epsilon^{-500n^2\pi^2t}. \tag{79}$$

The presence of the exponential terms in Eqs. (77) and (79) makes both series converge rapidly for $t > 0$, unless t is very small. For $t = 0$ and $0 < x < 100$, the series in Eq. (77) equals the initial values given by Eq. (66), and these values are approached by $e(x,t)$ as $t \to 0$. But for $t = 0$ and $0 < x < 100$, the series in Eq. (79) oscillates, and no values are approached by $i(x,t)$ as $t \to 0$. This indetermination of $i(x,0)$ is due to the discontinuity of our boundary conditions. For $t = 0$ and $x = 100$ the series in Eq. (79) does diverge to plus infinity, showing that $i(100,t)$ is very large for very small values of t.

Let us outline the method of solution in more general terms. Suppose that originally for a line AB, S miles long, the potential was some given function of the distance, $g(x)$, and that at time $t = 0$ the terminal A had its potential changed to r volts and that the terminal B had its potential changed to s volts, and that

both terminals were maintained at the new voltages. Here the boundary conditions are

$$e(x,0) = g(x), \qquad e(0,t) = r, \qquad e(S,t) = s. \qquad (80)$$

We transform this to a problem involving e_T by writing

$$e = e_S + e_T = \frac{s-r}{S} x + r + e_T, \qquad (81)$$

in which we have replaced e_S by an expression like that in Eq. (64) with constants such that the end conditions of Eq. (80) are satisfied. The boundary conditions for e_T then become

$$e_T(x,0) = g(x) - \frac{s-r}{S} x - r, \quad e_T(0,t) = 0, \quad e_T(S,t) = 0. \quad (82)$$

We may find e_T from a series like that in Eq. (74),

$$e_T(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{S} \, \epsilon^{-n^2\pi^2 t/RCS^2}, \qquad (83)$$

which satisfies Eq. (63) and the last two boundary conditions of Eq. (82). It will also satisfy the first condition of Eq. (82) if we replace the B_n in Eq. (83) by the coefficients b_n of the expansion of $f(x) = g(x) - \frac{s-r}{S} x - r$ in a Fourier sine series of half-period S. We determine these b_n from Eq. (10) or one of the alternatives to it mentioned in Sec. 39. We may then insert the values of B_n in

$$e(x,t) = \frac{s-r}{S} x + r + \sum_{n=1}^{\infty} B_n \cos \frac{n\pi x}{S} \, \epsilon^{-n^2\pi^2 t/RCS^2}. \qquad (84)$$

and in the relation obtained from it and Eq. (78)

$$i(x,t) = \frac{r-s}{RS} - \frac{\pi}{RS} \sum_{n=1}^{\infty} n B_n \cos \frac{n\pi x}{S} \, \epsilon^{-n^2\pi^2 t/RCS^2}. \qquad (85)$$

Equations (84) and (85) then give the solution of Eqs. (62) and the boundary conditions (80).

EXERCISE XXIII

The length of an ocean cable is $3{,}142 = 1{,}000\pi$ miles. The series resistance is 3 ohms per mile, the shunt capacitance is $\frac{1}{3}$ microfarad per mile or $\frac{1}{3} \times 10^{-6}$ farad per mile. Assuming that Eqs. (62) apply, find $e(x,t)$ and $i(x,t)$ where x miles is the distance from one end and t sec. is the time after both ends were suddenly grounded if initially the potential was

1. $e(x,0) = E \sin \dfrac{x}{500}.$ **2.** $e(x,0) = E_1 \sin \dfrac{x}{1{,}000} + E_{10} \sin \dfrac{x}{100}.$

3. $e(x,0) = E.$ **4.** $e(x,0) = \dfrac{Ex}{1{,}000\pi}.$

A telegraph cable is S miles long. Assuming that Eqs. (62) apply, find $e(x,t)$ and $i(x,t)$ t sec. after

5. The ends are grounded, if initially $e(x,0) = Ex/S$, the steady-state condition due to one end being grounded and the other at the constant potential E.

6. The ends are grounded, if initially $e(x,0) = E$, a constant potential.

7. The end $x = 0$ is grounded and the end $x = S$ is suddenly connected to a constant potential E, if initially $e(x,0) = 0$.

8. For the cable of Prob. 7, show that the current at the receiving end, $x = 0$, is given by

$$i(0,t) = -\frac{E}{RS}\left[1 - 2(\epsilon^{-\pi^2 t/S^2 RC} - \epsilon^{-4\pi^2 t/S^2 RC} + \epsilon^{-9\pi^2 t/S^2 RC} - \cdots)\right].$$

9. Jacobi's theta function $\theta(z,q)$ is defined by

$$\theta(z,q) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz$$

Some authors use θ_4 or θ_0 in place of θ. Show that in Prob. 8

$$i(0,t) = -\frac{E}{RS}\,\theta(0,\epsilon^{-\pi^2 t/S^2 RC}).$$

10. It is shown in the theory of theta functions, or from the Poisson sum formula in the theory of Fourier series and integrals that

$$\theta(0,\epsilon^{-u}) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n \epsilon^{-n^2 u} = 2 \sqrt{\frac{\pi}{u}} \sum_{n=1}^{\infty} \epsilon^{-(2n-1)^2 \pi^2/4u}.$$

Verify that the two sides are equal for $u = 1$. This transformation is useful in computing $i(0,t)$ of Probs. 8 and 9 for t near 0.

Use the data given in connection with Probs. 1 to 4, and the result of Prob. 8 combined with the equations in Probs. 9 and 10 where useful, to show that when a signal is sent the current received is

11. Approximately 90 per cent of its maximum value after 3 sec.
12. Approximately 73 per cent of its maximum at the end of 2 sec.
13. Approximately 30 per cent of its maximum at the end of 1 sec.
14. Less than 0.0004 of its maximum after $\frac{1}{4}$ sec.

15. Use the defining equation for $\theta(z,q)$ in Prob. 9 to show that for the cable of Prob. 7 the current at any point x after time t may be written

$$i(x,t) = -\frac{E}{RS} \theta\left(\frac{\pi x}{2S}, \epsilon^{-\pi^2 t/S^2 R C}\right).$$

Graphs, brief tables, and references to literature on theta functions are given in Jahnke-Emde's *Tables of Functions*. But it is necessary to be familiar with the meaning of several parameters and to make considerable preliminary computations in order to obtain a particular numerical value of $\theta(z,q)$ from most of the existing tables.

43. The Vibrating String

The wave equation in one dimension,

$$\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2} \tag{86}$$

was introduced as Eq. (81) of Sec. 31, where we showed that its general solution represented traveling waves.   Equation (86) is fundamental in the study of wave motion.   As the discussion of Secs. 31, 33, and 36 indicates, problems involving mechanical vibrations, sound waves, electromagnetic waves, or the propagation of electricity along a lossless transmission line all lead to an equation of this form.

For definiteness, in this section we shall investigate problems suggested by the small vibrations of a tightly stretched string, such as that in a violin or other stringed musical instrument. Then as in Eq. (81) of Sec. 31 u is the small transverse displacement of a point originally at distance x along the string and $v^2 = gT/D$, where T is the tension, D the weight per unit length, and g the acceleration of gravity.   Thus v has the units of velocity, or of x/t.

In particular let us consider the string AB of length 100 units, and measure x from A as in Fig. 45.   Denote the displacement by

FIG. 45.

$u(x,t)$ and the transverse velocity by $u_t(x,t) = \dfrac{\partial u}{\partial t}$.   Then, if the ends are fixed,

$$u(0,t) = 0, \qquad u(100,t) = 0. \tag{87}$$

We assume that the initial displacement and velocity are given by

$$u(x,0) = 12 \sin \frac{\pi x}{50}, \qquad u_t(x,0) = 5 \sin \frac{\pi x}{25}. \tag{88}$$

We wish to determine the subsequent motion, that is, the solution

$u(x,t)$ for $t > 0$ of Eq. (86) which satisfies the boundary conditions (87) and the initial conditions (88).

It follows from Prob. 5 of Exercise XVII that Eq. (86) admits

$$u = (c_1 \sin kx + c_2 \cos kx)(c_3 \sin kvt + c_4 \cos vt) \qquad (89)$$

as a particular solution of the form $X(x) \cdot T(t)$. The first condition (87) will be satisfied if $c_2 = 0$, and the second will be satisfied if

$$\sin 100k = 0, \qquad 100k = n\pi, \qquad \text{or} \qquad k = \frac{n\pi}{100}, \qquad (90)$$

where n is any positive integer. Let us write C_n for the product $c_1 c_4$ which goes with a particular n, and D_n for the product $c_1 c_3$ which goes with a particular n. Then we have

$$\left(C_n \cos \frac{n\pi vt}{100} + D_n \sin \frac{n\pi vt}{100} \right) \sin \frac{n\pi x}{100}, \qquad n = 1, 2, 3, \cdots \qquad (91)$$

as a set of terms each of which satisfies Eq. (86) and the boundary conditions of Eq. (87). Hence we put

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi vt}{100} + D_n \sin \frac{n\pi vt}{100} \right) \sin \frac{n\pi x}{100}. \qquad (92)$$

The initial conditions (88) will be satisfied, provided that

$$u(x,0) = 12 \sin \frac{\pi x}{50} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{100}, \qquad (93)$$

and

$$u_t(x,0) = 5 \sin \frac{\pi x}{25} = \sum_{n=1}^{\infty} \frac{n\pi v}{100} D_n \sin \frac{n\pi x}{100}. \qquad (94)$$

This suggests that we expand the given values of $u(x,0)$ and $u_t(x,0)$ in Fourier sine series of frequency $\pi/100$, and half-period 100. In this case we may use the procedure which led to Eqs. (17) and (19). We observe that $\pi/50 = 2\pi/100$ and

$$\frac{\pi}{25} = \frac{4\pi}{100},$$

each an integral multiple of the desired frequency $\pi/100$. Hence Eq. (93) will be satisfied if $C_2 = 12$, and $C_n = 0$ if $n \neq 2$. And Eq. (94) will hold if

$$5 = \frac{4\pi v}{100} D_4, \quad \text{or} \quad D_4 = \frac{125}{\pi v} \quad \text{and} \quad D_n = 0, \quad \text{if } n \neq 4.$$

On putting the values of C_n and D_n just found in Eq. (92), we have

$$u(x,t) = 12 \cos \frac{\pi v t}{50} \sin \frac{\pi x}{50} + \frac{125}{\pi v} \sin \frac{\pi v t}{25} \sin \frac{\pi x}{25} \quad (95)$$

as the solution of our problem.

Let us outline the method of solution in more general terms. Suppose that the string is S units long and that the ends are fixed so that

$$u(0,t) = 0, \qquad u(S,t) = 0. \quad (96)$$

And let the initial displacement and velocity be given by

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x). \quad (97)$$

Our problem is to determine the subsequent motion, that is, to find the solution $u(x,t)$ for $t > 0$ of Eq. (86) which satisfies the boundary conditions, Eqs. (96), and the initial conditions, Eqs. (97).

In place of Eq. (92) in this case we now have

$$u(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos \frac{n\pi v t}{S} + D_n \sin \frac{n\pi v t}{S} \right) \sin \frac{n\pi x}{S}, \quad (98)$$

which satisfies Eqs. (86) and (96). It will also satisfy Eq. (97) if

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{S} \quad (99)$$

and

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi v}{S} D_n \sin \frac{n\pi x}{S}. \quad (100)$$

Equation (99) shows that the C_n are the b_n of Eq. (9) with $L = S$. These may be determined from Eq. (10) with $L = S$ or one of the alternatives to it mentioned in Sec. 37. And from Eq. (100) we have

$$b_n = \frac{n\pi v}{S} D_n \quad \text{and} \quad D_n = \frac{S}{n\pi v} b_n, \qquad (101)$$

where here the b_n are those of Eq. (9) with $f(x) = g(x)$ and $L = S$. These coefficients of the expansion of $g(x)$ in a Fourier sine series of half-period S are also found by one of the methods of Sec. 37.

After calculating the C_n and D_n, substitution of their values in Eq. (98) leads to the solution of Eq. (86) which satisfies the conditions of Eqs. (96) and (97).

EXERCISE XXIV

Find the displacement $u(x,t)$ of a tightly stretched string of length S units vibrating between fixed end points if the initial velocity was zero and the initial displacement $u(x,0)$ was

1. $2 \sin \dfrac{\pi x}{S}.$ **2.** $p \sin^3 \dfrac{\pi x}{S}.$ **3.** $pSx - px^2.$

4. $p \sin \dfrac{k\pi x}{S}$, where k is some particular positive integer.
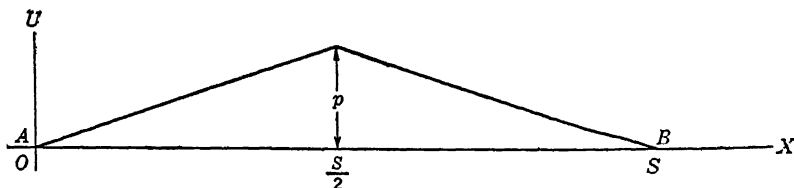
FIG. 46. $u(x,0)$ for the plucked string.

5. $\dfrac{2px}{S}$, if $0 < x < \dfrac{S}{2}$, and $2p - \dfrac{2px}{S}$, if $\dfrac{S}{2} < x < S$. This displacement, Fig. 46, approximates that for a *plucked* musical string.

Find the displacement $u(x,t)$ of a tightly stretched string of length S units vibrating between fixed end points if the string was initially in the equilibrium position, $u(x,0) = 0$, and the initial velocity $\dfrac{\partial u}{\partial t}$ at $t = 0$ or $u_t(x,0)$ was

6. $5 \sin \dfrac{3\pi x}{S}.$ **7.** $q \sin^3 \dfrac{\pi x}{S}.$ **8.** $qSx - qx^2.$

9. $q \sin \dfrac{k\pi x}{S}$, where k is some particular positive integer.

10. 0, if $0 < x < \dfrac{S-w}{2}$; q, if $\dfrac{S-w}{2} < x < \dfrac{S+w}{2}$;

$$0, \text{ if } \dfrac{S+w}{2} < x < S.$$

This condition approximates that for a *hammered* musical string.

11. Find the displacement $u(x,t)$ of a tightly stretched string of length S units vibrating between fixed end points if initially the displacement $u(x,0) = 3 \sin \dfrac{2\pi x}{S}$ and the velocity

$$u_t(x,0) = 4 \sin \dfrac{5\pi x}{S}.$$

12. Let $f(x)$ be defined for $0 < x < S$, and $F(x)$ be an odd function of period $2S$ which equals $f(x)$ on the interval $0,S$. Thus $F(x) = f(x)$, $0 < x < S$; $F(-x) = F(x)$; $F(x + 2S) = F(x)$. Show that if Eq. (99) holds, for $0 < x < S$, then its right member equals $F(x)$ for all values of x. Replace x by $x + vt$ and then by $x - vt$ and deduce that

$$F(x + vt) + F(x - vt) = \sum_{n=1}^{\infty} C_n \left[\sin \frac{n\pi(x + vt)}{S} \right.$$

$$\left. + \sin \frac{n\pi(x - vt)}{S} \right]$$

$$= \sum_{n=1}^{\infty} 2C_n \cos \frac{n\pi vt}{S} \sin \frac{n\pi x}{S}.$$

13. Using Prob. 12, show that for any one of Probs. 1 to 5 the solution may be written in the form

$$u(x,t) = \tfrac{1}{2}[F(x + vt) + F(x - vt)],$$

where $f(x) = u(x,0)$ as given for $0 < x < S$, and $F(x)$ is obtained from $f(x)$ as described in Prob. 12.

14. By reasoning like that used for the general solution of Eq. (81) in Sec. 31, show that the position of the string in **Prob. 13** at any time t can be found by moving the curve $u = \tfrac{1}{2}F(x)$ to the right a distance vt and moving an identical curve to the left a distance vt, and then adding the ordinates of the two curves in the interval $0,S$.

15. Apply Probs. 13 and 14 to the plucked string of Prob. 5.
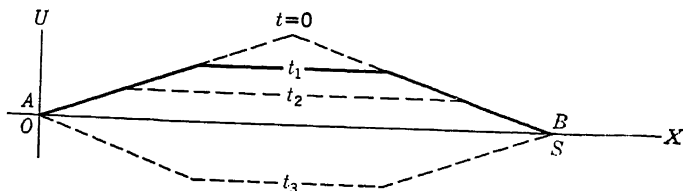
Fig. 47. $u(x,t)$ for the plucked string.

Hence show that, as in Fig. 47, the general position is an isosceles trapezoid.

16. Let $g(x)$ be defined for $0 < x < S$, and $G(x)$ be an odd function of period $2S$ which equals $g(x)$ on the interval $0,S$. Thus $G(x) = g(x), 0 < x < S; G(-x) = G(x); G(x + 2S) = G(x)$. Show that if Eq. (100) holds, for $0 < x < S$, then its right member equals $G(x)$ for all values of x. Replace x by z, and integrate with respect to z from $x - vt$ to $x + vt$ to show that

$$\int_{x-vt}^{x+vt} G(z)dz = \sum_{n=1}^{\infty} vD_n \left[\cos \frac{n\pi(x + vt)}{S} - \cos \frac{n\pi(x - vt)}{S} \right]$$

$$= \sum_{n=1}^{\infty} 2vD_n \sin \frac{n\pi vt}{S} \sin \frac{n\pi x}{S}.$$

17. Using Prob. 16, show that for any one of Probs. 6 to 10 the solution may be written in the form

$$u(x,t) = \frac{1}{2v} \int_{x-vt}^{x+vt} G(z)dz,$$

where $g(x) = u_t(x,0)$ as given for $0 < x < S$, and $G(x)$ is obtained from $g(x)$ as described in Prob. 16.

18. Let $F(x)$ be related to the $f(x)$ of Eq. (99) as in Probs. 12 and 13 and let $G(x)$ be related to the $g(x)$ of Eq. (100) as in Probs. 16 and 17. Show that the solution of Eq. (86) satisfying the conditions (96) and (97) found from Eq. (98) is equivalent to

$$u(x,t) = \frac{1}{2}[F(x+vt) + F(x-vt)] + \frac{1}{2v} \int_{x-vt}^{x+vt} G(z)dz.$$

19. Apply Prob. 17 to the hammered string of Prob. 10, at its mid-point $x = S/2$. In particular, show that $u\left(\frac{S}{2},t\right) = qt$ if $0 < t < \frac{w}{2v}$ and $u\left(\frac{S}{2},t\right) = \frac{qw}{2v}$ if $\frac{w}{2v} < t < \frac{3S}{2v} - \frac{w}{2v}$.

44. The Lossless Transmission Line

In Eq. (97) of Sec. 33 we showed that the equations governing the propagation of potential $e(x,t)$ and current $i(x,t)$ along a lossless transmission line having a series inductance of L henrys per mile and a shunt capacitance of C farads per mile are the *radio equations*

$$-\frac{\partial e}{\partial x} = L\frac{\partial i}{\partial t}, \qquad -\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t}, \qquad \frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2}. \quad (102)$$

The third relation is equivalent to

$$\frac{\partial^2 e}{\partial t^2} = v^2\frac{\partial^2 e}{\partial x^2}, \qquad \text{if } v = \frac{1}{\sqrt{LC}}. \quad (103)$$

This has the same form as Eq. (86). When boundary conditions for e are known, Eq. (103) may be solved by the method of Sec.

41.   And after e has been found, the first two relations of Eq. (102) determine i to within an additive constant of integration.

As an example, let us consider a line AB 100 miles long for which $L = 2 \times 10^{-4}$ henry per mile and $C = 2$ microfarads per mile.   Suppose that the initial potential was given by

$$e(x,0) = E \sin \frac{\pi x}{100}, \tag{104}$$

and that the initial current was constant

$$i(x,0) = I_0. \tag{105}$$

And as indicated in Fig. 48, suppose that at the instant taken as
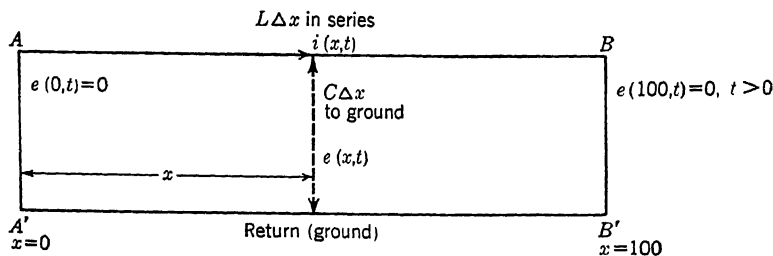
FIG. 48.

$t = 0$ the two ends of the line A and B were suddenly grounded. Then

$$e(0,t) = 0, \qquad e(100,t) = 0. \tag{106}$$

We wish to find the potential $e(x,t)$ and the current $i(x,t)$ at any point of the line $0 < x < 100$, and at any time after the grounding, $t > 0$.

We assume that the initial state was the result of a physical situation.   Then Eqs. (102) hold, and from the second relation

$$e_t(x,0) = \frac{\partial e}{\partial t}\bigg|_{t=0} = -\frac{1}{C}\frac{\partial i}{\partial x}\bigg|_{t=0} = -\frac{1}{C}\frac{\partial i(x,0)}{\partial x}. \tag{107}$$

But from Eq. (105), the last derivative is $\dfrac{\partial I_0}{\partial x} = 0$ so that

$$e_t(x,0) = 0. \tag{108}$$

Our boundary value problem for the potential is to determine the solution $e(x,t)$ for $t > 0$ of Eq. (103) which satisfies the boundary conditions (106) and the initial conditions (104) and (108). The similarity of Eqs. (103) and (106) to Eqs. (86) and (87) leads us to examine particular solutions of the form (91). If $D_n = 0$ these will also satisfy the condition (108). Thus we write

$$e(x,t) = \sum_{n=1}^{\infty} C_n \cos \frac{n\pi vt}{100} \sin \frac{n\pi x}{100}. \tag{109}$$

This satisfies Eqs. (103), (106), and (108). It will also satisfy the remaining condition (104), provided that

$$e(x,0) = E \sin \frac{\pi x}{100} = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{100}. \tag{110}$$

Since the given sine term has the same frequency as the series, $\pi/100$, we take $C_1 = E$ and the remaining $C_n = 0$, $n \neq 1$. Hence (109) becomes

$$e(x,t) = E \cos \frac{\pi vt}{100} \sin \frac{\pi x}{100}. \tag{111}$$

To find $i(x,t)$ we use Eq. (102). The first relation gives:

$$\frac{\partial i}{\partial t} = -\frac{1}{L} \frac{\partial e}{\partial x} = -\frac{E\pi}{100L} \cos \frac{\pi vt}{100} \cos \frac{\pi x}{100}. \tag{112}$$

The result of integrating both sides with respect to t is

$$i(x,t) = -\frac{E}{vL} \sin \frac{\pi vt}{100} \cos \frac{\pi x}{100} + f(x). \tag{113}$$

Here $f(x)$ is an arbitrary function of x, since x was kept constant during the integration. Now substitute the value of $\frac{\partial i}{\partial x}$ as obtained from Eq. (113) and the value of $\frac{\partial e}{\partial t}$ as obtained from

Eq. (111) in the second relation of Eq. (102). By observing that the definition of v in Eq. (103) implies that $v^2 = 1/(LC)$ or $Cv = 1/(vL)$, one may reduce the relation to $f'(x) = 0$ which shows that $f(x)$ must be a constant. And Eq. (105) shows that the constant must be I_0.

Hence

$$i(x,t) = -\frac{E}{vL} \sin \frac{\pi vt}{100} \cos \frac{\pi x}{100} + I_0. \qquad (114)$$

Finally we recall that $L = 2 \times 10^{-4}$ and $C = 2 \times 10^{-6}$, so that $v = 1/\sqrt{LC} = 5 \times 10^4$, $v/100 = 500$, $vL = 10$. Hence Eqs. (111) and (114) give

$$e(x,t) = E \cos 500\pi t \sin \frac{\pi x}{100},$$

$$i(x,t) = -\frac{E}{10} \sin 500\pi t \cos \frac{\pi x}{100}, \qquad (115)$$

as the solution of our problem.

Let us outline the method of solution for a more general case. Suppose that originally for a line AB, S miles long, the potential was a given function of the distance $f(x)$, and the current was another such function $g(x)$. And assume that at time $t = 0$ the terminal A had its potential changed to r volts and that the terminal B had its potential changed to s volts and that both terminals were maintained at the new voltages. Here the boundary conditions are

$$e(x,0) = f(x), \quad i(x,0) = g(x), \quad e(0,t) = r, \quad e(S,t) = s. \quad (116)$$

As in Eq. (81) we transform this to a problem involving e_T by writing

$$e = e_s + e_T = \frac{s-r}{S} x + r + e_T. \qquad (117)$$

The boundary conditions for e_T then become

$$e_T(x,0) = f(x) - \frac{s-r}{S} x - r, \quad e_T(0,t) = 0, \quad e_T(S,t) = 0, \quad (118)$$

obtained directly from Eqs. (116) and (117), together with

$$e_{T,t}(x,0) = \frac{\partial e_T}{\partial t}\Big|_{t=0} = e_t(x,0) = -\frac{1}{C}g'(x), \qquad (119)$$

obtained from Eqs. (117) and (107) and from $i(x,0) = g(x)$. We may find e_T from a series like that in Eq. (98),

$$e_T(x,t) = \sum_{n=1}^{\infty} \left(C_n \cos\frac{n\pi vt}{S} + D_n \sin\frac{n\pi vt}{S} \right) \sin\frac{n\pi x}{S}. \quad (120)$$

This satisfies the end conditions in Eq. (118). Equations similar to Eqs. (99) and (100) show that the C_n must be the coefficients of the expansion of $f(x) - \frac{s-r}{S}x - r$ in a Fourier sine series of half-period S to satisfy the first condition of Eq. (118) and that the D_n must be found from Eq. (101) from b_n, the coefficients of the expansion of $-\frac{1}{C}g'(x)$ in a Fourier sine series of half-period S to satisfy the condition of Eq. (119). The values so found may be inserted in

$$e(x,t) = \frac{s-r}{S}x - r$$
$$+ \sum_{n=1}^{\infty} \left(C_n \cos\frac{n\pi vt}{S} + D_n \sin\frac{n\pi vt}{S} \right) \sin\frac{n\pi x}{S} \quad (121)$$

to give the solution for $e(x,t)$.

Use of the first two relations of Eqs. (102) in a manner similar to that which leads from Eqs. (111) to (114) shows that the $i(x,t)$ which corresponds to the $e(x,t)$ of Eq. (121) is given by

$$i(x,t) = -\frac{1}{L}\frac{s-r}{S}t + a$$
$$-\frac{1}{vL}\sum_{n=1}^{\infty} \left(C_n \sin\frac{n\pi vt}{S} - D_n \cos\frac{n\pi vt}{S} \right) \cos\frac{n\pi x}{S}, \quad (122)$$

where

$$v = \frac{1}{\sqrt{LC}} \quad \text{and} \quad a = \frac{1}{S} \int_0^S g(x)dx, \tag{123}$$

so that a is the constant term in the expansion of $g(x)$ in a Fourier cosine series of half-period S.

EXERCISE XXV

A transmission line is S miles long, and it is lossless so that Eqs. (102) apply. Initially $i(x,0) = I_0$ so that $e_t(x,0) = 0$. Find $e(x,t)$ and $i(x,t)$ t sec. after both ends were suddenly grounded if initially the potential was

1. $e(x,0) = E \sin \frac{\pi x}{S}$.

2. $e(x,0) = E_2 \sin \frac{2\pi x}{S} + E_5 \sin \frac{5\pi x}{S}$.

3. $e(x,0) = E$.

4. $e(x,0) = \frac{Ex}{S}$.

5. A lossless transmission line of length S is initially uncharged, so that $i(x,0) = 0$, $e(x,0) = 0$, $e_t(x,0) = 0$. At $t = 0$ the end $x = S$ is suddenly connected with a constant potential E while the other end is grounded. Thus $e(S,t) = E$, $e(0,t) = 0$. Use Eq. (102) and find $e(x,t)$ and $i(x,t)$ for $t > 0$.

6. A lossless transmission line is S miles long. The end $x = 0$ is grounded so that $e(0,t) = 0$. The other end is left open, so that $i(S,t) = 0$ and $e_x(S,t) = -Li_t(S,t) = 0$. Show that for $n = 1, 3, 5, \cdots$ any odd integer, $B_n \sin \frac{n\pi x}{2S} \cos \frac{n\pi vt}{2S}$ is a particular solution of Eq. (103) satisfying the end condition.

7. Suppose that just before the end conditions of Prob. 6 were imposed, the line had $e(x,0) = Ex/S$. Find $e(x,t)$ by using a series of particular solutions of Prob. 6 and an expansion in an odd-harmonic sine series similar to that used in Prob. 20 of Exercise XXII.

8. A lossless transmission line of length S is initially uncharged, so that $i(x,0) = 0$, $e(x,0) = 0$, $e_t(x,0) = 0$. At $t = 0$, the end $x = 0$ is suddenly connected with a constant potential E,

while the other end $x = S$ is left open as in Prob. 6, so that $i(S,t) = 0$, $e_x(S,t) = 0$. Find $e(x,t)$ and hence $i(x,t)$. HINT: Put $e(x,t) = E + e_t(x,t)$, and determine $e_T(x,t)$ by the method used in Prob. 7.

9. Show that for any one of Probs. 1 to 4 the solution for the potential may be written in the form

$$e(x,t) = \tfrac{1}{2}[F(x + vt) + F(x - vt)],$$

where $f(x) = e(x,0)$ as given for $0 < x < S$, and $F(x)$ is related to $f(x)$ as in Probs. 12 and 13 of Exercise XXIV.

10. If the emf for a line of length S satisfies Eq. (103) and the conditions $e(x,0) = 0$, $e_t(x,0) = g(x)$, $e(0,t) = 0$, $e(S,t) = 0$ show that the potential may be written in the form

$$e(x,t) = \frac{1}{2v} \int_{x-vt}^{x+vt} G(z)dz$$

where $G(x)$ is related to $g(x)$ as in Probs. 16 and 17 of Exercise XXIV.

A line is 50 miles long, has series resistance $R = 0.12$ ohms per mile, series inductance $L = 2 \times 10^{-3}$ henry per mile, conductance to ground $G = \tfrac{4}{3} \times 10^{-8}$ mho per mile, and shunt capacitance to ground $C = 1.2 \times 10^{-8}$ farad per mile. Hence Eqs. (92), (93) and (94) of Sec. 33 must be used. For this line

11. Show that $i = I_0 \epsilon^{0.00004x}$, $e = 3,000 I_0 \epsilon^{0.00004x}$ are possible steady-state values.

12. Show that if $e(0,t) = 0$ and $e(50,t) = 0$, then Eq. (93) of Sec. 33 admits $C_n \sin \dfrac{n\pi x}{50} \epsilon^{-30.3t} \left(\dfrac{30.3}{\beta_n} \sin \beta_n t + \cos \beta_n t \right)$ as particular solutions satisfying the end conditions if

$$\beta_n = \sqrt{\frac{n^2 \pi^2}{6} - 851}$$

and n is a positive integer.

13. If the initial values $i(x,0)$ and $e(x,0)$ are those given in Prob. 11 so that $e_t(x,0) = 0$, and the ends are suddenly grounded,

find the emf $e(x,t)$ by using a series of the particular solutions found in Prob. 12.

14. If the line is initially uncharged $e(x,0) = 0$, $i(x,0) = 0$, $e_t(x,0) = 0$, $i_t(s,0) = 0$. Find $e(x,t)$ if at $t = 0$ the end $x = 50$ has a constant potential E suddenly impressed on it and the other end is suddenly grounded. Note that the steady-state solution for the permanent end conditions has the form $c_1\epsilon^{\sqrt{RGx}} + c_2\epsilon^{-\sqrt{RGx}}$, and with constants chosen to take on the proper end values is

$$e_S = E\frac{\epsilon^{0.00004x} - \epsilon^{-0.00004x}}{\epsilon^{0.002} - \epsilon^{-0.002}} = \frac{E \sinh 0.00004x}{\sinh 0.002}$$

Put $e = e_S + e_T$, and find e_T by the method of Prob. 13.

45. Hollow Wave Guides

The discussion of Maxwell's equations in Secs. 34 and 36 showed that in a homogeneous isotropic nonconducting medium free of electric charge, such as the space inside a copper pipe, an electromagnetic field may be mathematically characterized by the electric field intensity vector **E,** with scalar components E_x, E_y, E_z and the magnetic field intensity vector **H,** with scalar components H_x, H_y, H_z. There are four fundamental vector relations which **E** and **H** must satisfy.

We shall restate here in terms of the components those equations implied by Maxwell's laws which are useful in studying the fields inside hollow wave guides. It follows from Eq. (117) of Sec. 36 that each of the six scalar components E_x, E_y, E_z, H_x, H_y, H_z satisfies a *wave equation*

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \qquad (124)$$

The *continuity equation* for E,

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0, \qquad (125)$$

follows from Eq. (114) of Sec. 36 and Eq. (103) of Sec. 34. And the time derivatives of the components of magnitic field intensity

are related to the electric field intensity by the first relation of
Eq. (114) of Sec. 36, or Eq. (101) of Sec. 34, namely,

$$-\mu_0 \frac{\partial H_x}{\partial t} = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z},$$

$$-\mu_0 \frac{\partial H_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, \qquad (126)$$

$$-\mu_0 \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}.$$

Consider a hollow conducting tube of constant rectangular
cross section, Fig. 49.   Let the sides of the rectangle be a and b
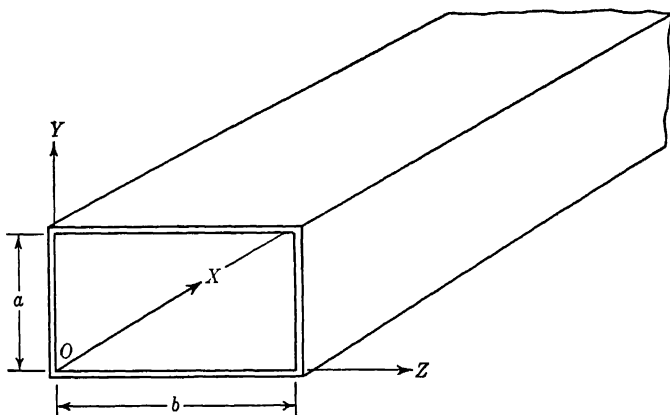
Fig. 49.   Hollow wave guide.   At the bounding faces **E** is parallel to OY for
$y = 0$ and $y = a$, **E** is parallel to OZ for $z = 0$ and $z = b$.

units in length parallel, respectively, to the y and z axes, and let
the x axis coincide with one longitudinal edge of the tube.   Thus
the plane faces of the tube lie in the planes $y = 0$, $y = a$, $z = 0$,
and $z = b$.

We assume that the tube is a perfect conductor.   Hence the
components of **E** tangent to the surface of the tube must be zero.
That is, in our case,

$$E_x = 0 \quad \text{and} \quad E_y = 0 \quad \text{if } z = 0 \text{ or if } z = b$$
$$E_x = 0 \quad \text{and} \quad E_z = 0 \quad \text{if } y = 0 \text{ or if } y = a. \quad (127)$$

The discussion in connection with Eqs. (81) and (82) of Sec. 31 showed that $f(x - vt)$ corresponded to wave motion in the positive x direction. The same will be true of any function of $(\omega t - \beta x)$ since

$$\omega t - \beta x = -\beta(x - vt) \text{ if } v = \frac{\omega}{\beta}. \tag{128}$$

In order to restrict our problem, we assume that the wave form is sinusoidal. Then each component of **E** and **H** will have the form

$$u = U(y,z)[c_1 \cos (\omega t - \beta x) + c_2 \sin (\omega t - \beta x)]. \tag{129}$$

This will be solution of Eq. (124), provided that

$$\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = -k^2 U, \quad \text{where } k^2 = \frac{\omega^2}{c^2} - \beta^2. \tag{130}$$

Since all the relations are linear, other wave forms could be built up by combining solutions of the type of Eq. (129) for different ω in finite sums, Fourier series, or Fourier integrals. Here we shall confine our attention to a single frequency. Then ω is given, subject to certain restrictions which we shall derive presently, and β or k is a constant chosen so as to satisfy the boundary conditions.

The restricted problem requires us to find three solutions of Eq. (130) which may be combined with pairs of constants in Eq. (129) to give values E_x, E_y, E_z, which satisfy the boundary conditions, Eq. (127), and Eq. (125). The boundary conditions (127) suggest the use of products of trigonometric functions, and the seeking of solutions of Eq. (130) of the form $Y(y) \cdot Z(z)$ as in Sec. 30. We begin by noting that

$$U(y,z) = \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} ry \begin{Bmatrix} \sin \\ \cos \end{Bmatrix} sz \tag{131}$$

will be a solution of Eq. (130) if $r^2 + s^2 = k^2$. To satisfy Eq. (127), we use $\sin ry$ in E_x and E_z with $r = m\pi/a$ and we use $\sin sz$ in E_x and E_y with $s = n\pi/b$. Here m and n are each zero or a positive integer. To make possible a cancellation of the

derivatives in Eq. (125), use use $\cos ry$ in E_y and $\cos sz$ in E_z. This leads us to write

$$E_x = \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b} [A_x \cos (\omega t - \beta x) + B_x \sin (\omega t - \beta x)],$$

$$E_y = \cos \frac{m\pi y}{a} \sin \frac{n\pi z}{b} [A_y \cos (\omega t = \beta x) + B_y \sin (\omega t - \beta x)],$$

$$E_z = \sin \frac{m\pi y}{a} \cos \frac{n\pi z}{b} [A_z \cos (\omega t - \beta x)$$
$$+ B_z \sin (\omega t - \beta x)]. \quad (132)$$

For any values of the constant coefficients A_x, B_x, A_y, B_y, A_z, B_z these will satisfy the conditions (127) if

$$m,n = 0, 1, 2, 3, \cdot \cdot \cdot . \quad (133)$$

And E_x, E_y, E_z will each satisfy the wave equation, Eq. (124), if $r^2 + s^2 = k^2$, or from Eq. (130) if

$$\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) = k^2 = \frac{\omega^2}{c^2} - \beta^2. \quad \text{Hence}$$

$$\beta^2 = \frac{\omega^2}{c^2} - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right). \quad (134)$$

To make β real, the last right member in Eq. (134) must be positive or zero. This requires that for any pair m,n

$$\omega^2 \geqq \pi^2 c^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \qquad \text{and}$$

$$\beta = \frac{\omega}{c} \sqrt{1 - \frac{\pi^2 c^2}{\omega^2} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}. \quad (135)$$

Since $m = 0$, $n = 0$ would make all three components in Eq. (132) equal to zero, for a nonzero solution the smallest value of ω results when we take $m = 0$, $n = 1$ or $n = 0$, $m = 1$. Thus if

$$b > a, m = 0, n = 1, \beta = 0 \qquad \text{and} \qquad \omega = \frac{\pi c}{b} \quad (136)$$

is the least possible value of the frequency ω.

The condition (125) will be satisfied if

$$-A_x - \frac{m\pi}{a} B_y - \frac{n\pi}{b} B_z = 0$$

$$B_x - \frac{m\pi}{a} A_y - \frac{n\pi}{b} A_z = 0. \tag{137}$$

These may be solved for two of the constants, for example, A_z and B_z, in terms of the remaining four. Some particular cases are worked out in more detail in the problems of Exercise XXVI. These problems also illustrate the use of Eq. (126) to determine $\mathbf{H}$.

As we indicated in Prob. 15 of Exercise XIX, when the wave guide is a round pipe, or hollow tube of circular cross section, it is convenient to use polar coordinates in the plane of cross section and the particular solutions for a given frequency involve Bessel functions.

46. References

Additional illustrations of the use of series of particular solutions to solve boundary value problems will be found in many of the books on heat, vibrations, sound, and electricity such as those already referred to in Secs. 14 and 37.

An elementary, but more complete, treatment which deduces the continuity of some of our solutions from Abel's theorem will be found in the first two texts mentioned in Sec. 25.

One method of justifying the validity and uniqueness of the series solutions of boundary value problems rests on the theory of integral equations. This will be found in such comprehensive works as Frank and von Mises, *Differential- und Integralgleichungen der Mechanik und Physik*, or Courant-Hilbert, *Methoden der mathematischen Physik*.

<div align="center">EXERCISE XXVI</div>

1. Let $A_x = 0$, $A_y = 0$, $B_x = 0$. Show that Eq. (137) implies that $A_z = 0$ when $n \neq 0$ and that it also determines the ratio of B_y to B_z. This leads to a particular solution which may

be written

$$E_x = 0,$$

$$E_y = -B\,\frac{\omega\mu_0}{k^2}\,\frac{n\pi}{b}\cos\frac{m\pi y}{a}\sin\frac{n\pi z}{b}\sin\,(\omega t - \beta x),$$

$$E_z = B\,\frac{\omega\mu_0}{k^2}\,\frac{m\pi}{a}\sin\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\sin\,(\omega t - \beta x).$$

Since **E** has no component in the axial or x direction, this is known as a *transverse electric* wave, or $TE_{m,n}$ wave.

2. If **E** is known, Eq. (126) determines H_x, H_y, H_z to within an additive arbitrary function of x, y, z for each component. These three functions must all be zero to make the components of **H** have the form of u in Eq. (129), as we assumed. Use these facts to show that for the $TE_{m,n}$ wave of Prob. 1, the magnetic field intensity vector **H** has components

$$H_x = -B\cos\frac{m\pi y}{a}\cos\frac{n\pi z}{b}\cos\,(\omega t - \beta x),$$

$$H_y = B\,\frac{\beta}{k^2}\,\frac{m\pi}{a}\sin\frac{m\pi y}{b}\cos\frac{n\pi z}{b}\sin\,(\omega t - \beta x),$$

$$H_z = B\,\frac{\beta}{k^2}\,\frac{n\pi}{b}\cos\frac{m\pi y}{a}\sin\frac{n\pi z}{b}\sin\,(\omega t - \beta x).$$

3. For the $TE_{m,n}$ wave let $\omega_{m,n}$ denote the *cutoff angular frequency* or smallest value of ω which leads to a real β. Show that

$$\omega_{m,n} = \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}.$$

If $\omega < \omega_{m,n}$ $\beta^2 < 0$. Hence β is imaginary and the field is rapidly attenuated in the x direction. Since c has the order of magnitude of 10^8, and a, b are comparable with unity, only ultrahigh frequencies $\omega > \omega_{m,n}$ can be propagated by wave guides of reasonable size.

4. Let $A_x = A$, $A_y = 0$, $B_x = 0$, and assume that

$$\frac{E_y}{E_z} = \frac{m}{n},\ n \neq 0.$$

From Eqs. (132) and (137) deduce that in this case

$$E_x = A \sin \frac{m\pi y}{a} \sin \frac{n\pi z}{b} \cos (\omega t - \beta x),$$

$$E_y = A \frac{\beta}{k^2} \frac{m\pi}{a} \cos \frac{m\pi y}{a} \sin \frac{nz}{b} \sin (\omega t - \beta x),$$

$$E_z = A \frac{\beta}{k^2} \frac{n\pi}{b} \sin \frac{m\pi y}{a} \cos \frac{nz}{b} \sin (\omega t - \beta x).$$

5. Use the procedure of Prob. 2 to show that the components of **H** which correspond to the vector **E** given in Prob. 4 are

$$H_x = 0,$$

$$H_y = -A \frac{K_0 \omega}{k^2} \frac{n\pi}{b} \sin \frac{m\pi y}{a} \cos \frac{n\pi z}{b} \sin (\omega t - \beta x),$$

$$H_z = A \frac{K_0 \omega}{k^2} \frac{m\pi}{a} \cos \frac{m\pi y}{a} \sin \frac{n\pi z}{b} \sin (\omega t - \beta x).$$

To reduce the coefficients to this form, note that by Eq. (116) of Sec. 36, $1/c^2 = \mu_0 k_0$, and hence from Eq. (134)

$$1 + \frac{\beta^2}{k^2} = \frac{\omega^2}{c^2 k^2} = \frac{\mu_0 K_0 \omega^2}{k^2}.$$

The condition $E_y/E_z = m/n$ in Prob. 4 was motivated by the desire to make $H_x = 0$. Since **H** has no component in the axial or x direction, the wave of Probs. 4 and 5 is known as a *transverse magnetic* wave, or $TM_{m,n}$.

6. Show that a more general solution with $E_x = 0$ equivalent to the form of Eq. (132) satisfying Eq. (137) than that of Prob. 1 results if we replace $(\omega t - \beta x)$ by $(\omega t - \beta x + \phi_1)$ in E_y, E_z as written in Prob. 1. The vector $\mathbf{H}_1$ which corresponds to this $\mathbf{E}_1$ is found by making the same substitution in H_x, H_y, H_z as written in Prob. 2.

7. Show that a more general solution $\mathbf{E}_2$, $\mathbf{H}_2$ with $H_x = 0$ than that of Probs. 4 and 5 where **E** and **H** are related by Eq. (126) and **E** is of the form (132) and satisfies Eq. (137) may be found by replacing $(\omega t - \beta x)$ by $(\omega t - \beta x + \phi_2)$ in the six components as written in Probs. 4 and 5.

8. For the vectors of Probs. 6 and 7 show that $\mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2$ is related to $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ by Eq. (126). Also show that $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$ is of the form of Eq. (132) and satisfies Eq. (137). As it involves four constants: A, B, ϕ_1, ϕ_2, it is the most general solution to be obtained from Eqs. (132) and (137).

9. For the electromagnetic fields of Probs. 1 and 2 or 6 show that

$$E_x = 0, \qquad E_y = -\mu_0 \frac{\omega}{\beta} H_z, \qquad E_z = \mu_0 \frac{\omega}{\beta} H_y.$$

10. For the electromagnetic fields of Probs. 4 and 5 or 7 show that

$$H_x = 0, \qquad H_y = -K_0 \frac{\omega}{\beta} E_z, \qquad H_z = K_0 \frac{\omega}{\beta} E_y.$$

11. From Probs. 9 and 10 deduce that for each type of electromagnetic field there considered

$$E_x H_x + E_y H_y + E_z H_z = \mathbf{E} \cdot \mathbf{H} = 0,$$

so that the vectors $\mathbf{E}$ and $\mathbf{H}$ are perpendicular.

CHAPTER 5

LAPLACE TRANSFORMS. TRANSIENTS

In Chaps. 1 and 3 we discussed a number of physical phenomena which were governed by a system of ordinary or partial linear differential equations. In many applications we seek the solution of such a linear system which satisfies certain initial conditions, or which takes on given values of the functions and their derivatives at time $t = 0$. For example, in studying the stability of a mechanical system it is helpful to know the response to a sudden jar, or to a periodic forcing term. And for an electric network we may wish to study the transient currents in a particular element when an electromotive force of given type is suddenly impressed on the input terminals.

For a simple system of one or two ordinary differential equations, each linear with constant coefficients, elementary methods could be applied. That is, we would first find the general solution in terms of arbitrary constants and then determine what value of the constants would fit the initial conditions. But for complicated systems the calculation of the solutions is greatly facilitated by using some form of *operational calculus*, or set of rules of procedure for translating the physical problem into a simplified system of equations from which we may directly calculate the solution which fits the initial conditions. Operational methods were initiated by an electrical engineer, Heaviside, and used by him to obtain a number of correct solutions of involved problems in electric circuit theory. Although some of Heaviside's original derivations were incomplete, it was later found possible to give sound proofs of his rules based on any one of several known mathematical theories, such as complex contour integration, integral equations, or the Laplace transformation.

Since the approach through Laplace transforms has proved to be the simplest and most comprehensive, we shall adopt it here. And we shall present the modern form of the operational calculus, differing in one minor point from Heaviside's form, which naturally follows from a consideration of the Laplace transform.

We first recall the definition of a Laplace transform and some of its properties which were given in Sec. 24, and derive some further properties. We then show how the transform converts a system of ordinary differential equations to an algebraic system and use this fact to solve certain electric and mechanical networks. Finally we show how the transform converts a system of partial differential equations to a system of ordinary differential equations and illustrate this procedure by some problems on the lossless long line.

47. The Laplace Transformation

Let t be a real variable, and $f(t)$ be any function given on the semiinfinite interval $0, +\infty$ and hence defined for all positive values of t. In many applications t is the time, and it is convenient to consider $f(t) = 0$ for $t < 0$. Then $F(p)$, the *Laplace transform* of $f(t)$, is defined by the equation

$$F(p) = \int_0^\infty e^{-pt} f(t) dt. \tag{1}$$

The new function $F(p)$ determined from $f(t)$ by the Laplace transformation does not involve the variable t, since the integration is between fixed limits. But it does involve the new variable p which was introduced in the exponential factor. Thus to each given function of t, Eq. (1) relates a transformed function of p. This relation may be expressed by writing

$$F(p) = \text{Lap } f(t) \qquad \text{or} \qquad F = \text{Lap } f. \tag{2}$$

When seeking the $f(t)$ which gave rise to $F(p)$, we may write

$$f(t) = \text{Lap}^{-1} F(p) \qquad \text{or} \qquad f = \text{Lap}^{-1} F. \tag{3}$$

In Eq. (2), Lap is read "the Laplace transform of." And in Eq. (3), Lap^{-1} is read "the inverse Laplace transform of."

Frequently the use of a small letter for the function of t and the corresponding capital for the transformed function of p is enough to indicate the relationship.   Thus in Eq. (123) of Sec. 24, we defined $G = $ Lap g, and discussed some of the properties of the relation in connection with Fourier transforms.   We pointed out that if g did not increase too rapidly for large positive values of t, or x in Eq. (117) of Sec. 24, and the real part of p exceeded some fixed positive value, the a of Eq. (121) of Sec. 24, then the integral defining the Laplace transform necessarily converged.   And the discussion of Eq. (125) of Sec. 24, which held for $g = $ Lap^{-1} G, showed that if g is made up of regular pieces it is uniquely determined by G inside any regular piece.

In this chapter all functions $f(t)$ of which we take the transforms will be piecewise regular, and will either be bounded or will increase less rapidly in numerical value than e^{at} for some positive a.   And all the functions $F(p)$ whose inverse transforms we seek will admit of such functions Lap^{-1} $F(p)$ satisfying the conditions imposed on $f(t)$.

In applying Eq. (1), we shall think of p as real, positive, and sufficiently large so that the integral on the right converges. We illustrate this process for some simple functions.

Suppose that $f(t) = 1$ for $t > 0$.   When we take $f(t) = 0$ for $t < 0$, this function is known as the *unit step*.   We find from Eq. (1) that

$$\text{Lap } 1 = \int_0^\infty 1 \cdot e^{-pt}\, dt = - \left.\frac{1}{p}\, e^{-pt}\right|_0^\infty$$
$$= \frac{1}{p}. \tag{4}$$

Let us next put $f(t) = e^{ct}$ in Eq. (1).   We find

$$\text{Lap } e^{ct} = \int_0^\infty e^{ct} e^{-pt}\, dt = - \left.\frac{1}{p-c}\, e^{-(p-c)t}\right|_0^\infty$$
$$= \frac{1}{p-c}. \tag{5}$$

Here we must take $p > c$ to make the integral finite at $t = +\infty$. The operation of forming the transform is linear in character.

In particular, it follows directly from our definition that

$$\text{Lap } [c_1 f(t)] = c_1 \text{ Lap } f(t) = c_1 F(p), \tag{6}$$

$$\text{Lap } [f(t) + g(t)] = \text{Lap } f(t) + \text{Lap } g(t) = F(p) + G(p), \tag{7}$$

$$\text{Lap } [c_1 f(t) + c_2 g(t)] = c_1 F(p) + c_2 G(p). \tag{8}$$

We may use these relations to find the transforms of cosh kt and sinh kt. From Eq. (35) of Sec. 4, we have

$$\cosh kt = \tfrac{1}{2}(e^{kt} + e^{-kt}), \qquad \sinh kt = \tfrac{1}{2}(e^{kt} - e^{-kt}). \tag{9}$$

By putting $c = k$, and $c = -k$ in Eq. (5), and using Eqs. (6) and (7), we find

$$\text{Lap } \cosh kt = \frac{1}{2}(\text{Lap } e^{kt} + \text{Lap } e^{-kt}) = \frac{1}{2}\left(\frac{1}{p-k} + \frac{1}{p+k}\right)$$

$$= \frac{p}{p^2 - k^2}. \tag{10}$$

$$\text{Lap } \sinh kt = \frac{1}{2}(\text{Lap } e^{kt} - \text{Lap } e^{-kt}) = \frac{1}{2}\left(\frac{1}{p-k} - \frac{1}{p+k}\right)$$

$$= \frac{k}{p^2 - k^2}. \tag{11}$$

Since p and t are real, it also follows from the definition that if i is the imaginary unit and if u, v, U, V are real, then

$$\text{Lap } (u + iv) = U + iV \qquad \text{implies that} \qquad \text{Lap } u = U$$
$$\text{and Lap } v = V. \tag{12}$$

Or, using the notation of Sec. 12, when f and F are complex functions of the real variables t and p, we have

$$\text{Lap } (\text{Re } f) = \text{Re } F \qquad \text{and} \qquad \text{Lap } (\text{Im } f) = \text{Im } F. \tag{13}$$

If we put $c = -a + ki$ in Eq. (5), the result may be written

$$\text{Lap } e^{-at+ikt} = \frac{1}{p+a-ki} = \frac{p+a+ki}{(p+a)^2 + k^2}. \tag{14}$$

But, from Eq. (28) of Sec. 4,

$$e^{-at+ikt} = e^{-at}\cos kt + ie^{-kt}\sin kt. \tag{15}$$

It follows from the last four equations that

$$\text{Lap } e^{-at} \cos kt = \frac{p + a}{(p + a)^2 + k^2}. \tag{16}$$

$$\text{Lap } e^{-at} \sin kt = \frac{k}{(p + a)^2 + k^2}. \tag{17}$$

48. Transforms of Derivatives

Suppose that we know $F(p)$, the transform of $f(t)$ and wish to find the transform of $f'(t)$, the derivative of $f(t)$. We assume that $f'(t)$ is piecewise regular, and that $f(t)$ is the integral of $f'(t)$ and hence continuous for all positive values of t. From the defining equation, Eq. (1), we have

$$\text{Lap } f'(t) = \int_0^\infty f'(t) e^{-pt}\, dt. \tag{18}$$

Integrating by parts, with $u = e^{-pt}$ and $dv = f'(t)dt$, $v = f(t)$, we find

$$\begin{aligned}
\text{Lap } f'(t) &= f(t)e^{-pt}\Big|_0^\infty + p \int_0^\infty f(t)e^{-pt}\, dt \\
&= -f(0+) + p \text{ Lap } f(t).
\end{aligned} \tag{19}$$

Here $f(0+)$ means the limit of $f(t)$ as t approaches zero through positive values.

As an illustration, let $f(t) = e^{ct}$. Then by Eqs. (19) and (5)

$$\text{Lap } ce^{ct} = -1 + p \frac{1}{p - c} = c \frac{1}{p - c}. \tag{20}$$

This is in agreement with Eqs. (5) and (6).

We may derive relations similar to Eq. (19) for higher derivatives, whenever the highest derivative considered is piecewise regular, and all lower derivatives are continuous for positive values of t. Thus for the second derivative, $f''(t)$, we first replace $f(t)$ by $f'(t)$ in Eq. (19) and so find

$$\text{Lap } f''(t) = -f'(0+) + p \text{ Lap } f'(t), \tag{21}$$

and then replace Lap $f'(t)$ in the right member by its value from Eq. (19). The result is

$$\text{Lap } f''(t) = -f'(0+) - pf(0+) + p^2 \text{ Lap } f(t). \tag{22}$$

Let us next obtain the transform of the integral

$$g(t) = \int_{t_0}^{t} f(t)dt = \int_{t_0}^{t} f(u)du. \tag{23}$$

We note that this makes $g'(t) = f(t)$. Then from Eq. (19), with f replaced by g throughout, we find

$$\text{Lap } f(t) = -g(0+) + p \text{ Lap } g(t). \tag{24}$$

Consequently,

$$\text{Lap } g(t) = \frac{\text{Lap } f(t) + g(0+)}{p} \tag{25}$$

In view of Eq. (23), this may be written

$$\text{Lap } \int_{t_0}^{t} f(u)du = \frac{\text{Lap } f(t) + \int_{t_0}^{0} f(u)du}{p} \tag{26}$$

As an application of this formula, consider the charge on a condenser which was discharged at time t_0,

$$q = \int_{t_0}^{t} i\, dt. \tag{27}$$

If q_0 denotes the charge at $t = 0$, or integral from t_0 to 0, from Eq. (26) we have

$$\text{Lap } q = \frac{\text{Lap } i + q_0}{p} \quad \text{or} \quad Q = \frac{I + q_0}{p} \tag{28}$$

49. Zero Initial Values

The relations of the preceding section take a simpler form if we assume that the initial value of the function, $f(0+)$, and of the first derivative $f'(0+)$ are each zero. In that case Eqs. (19) and (22) become

$$\text{Lap } f'(t) = p \text{ Lap } f(t), \quad \text{for } f(0+) = 0, \tag{29}$$

$$\text{Lap } f''(t) = p^2 \text{ Lap } f(t), \quad \text{for } f'(0+) = 0, f(0+) = 0. \tag{30}$$

Again, if we take $t_0 = 0$ in Eq. (26), it becomes

$$\text{Lap } \int_{0}^{t} f(u)du = \frac{1}{p} \text{ Lap } f(t). \tag{31}$$

The conditions $f(0+) = 0$, $f'(0+) = 0$ are met in mechanical systems if we start at rest, and choose the coordinate systems so that the initial displacements are zero. And in electrical systems dead for $t < 0$, all the condensers are discharged and the impressed emfs are zero for $t = 0$, so that if a current is determined by an nth-order differential equation, the initial value of the current and of each of its first $(n - 1)$ derivatives will be zero.

For such cases Eqs. (29) and (30) and the similar ones for higher derivatives show that in forming all the derivatives up to the nth we may consider differentiation of the function as corresponding to multiplication of the transform by p. And Eq. (31) shows that integration of the function with lower limit zero corresponds to division of the transform by p. Thus for the case of zero initial values the complicated operations of differentiation and integration are transformed into mere multiplication and division by p.

We shall apply Eq. (31) to find the transforms of the integral powers of t. Using Eq. (4), we find successively:

$$t = \int_0^t 1 \, dt, \qquad \text{Lap } t = \frac{1}{p} \text{Lap } 1 = \frac{1}{p^2}. \tag{32}$$

$$t^2 = 2 \int_0^t t \, dt, \qquad \text{Lap } t^2 = 2\frac{1}{p} \text{Lap } t = \frac{2}{p^3}. \tag{33}$$

$$t^n = n \int_0^t t^{n-1} \, dt, \qquad \text{Lap } t^n = n\frac{1}{p} \text{Lap } t^{n-1} = \frac{n!}{p^{n+1}}. \tag{34}$$

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$. If $n = 0$, $0! = 1$, $t^0 = 1$, and Eq. (34) reduces to Eq. (4).

50. Substitution and Translation Properties

Let us replace p by $p + a$ in the defining equation, Eq. (1). We thus find

$$\begin{aligned}
F(p + a) &= \int_0^\infty e^{-(p+a)t} f(t) dt \\
&= \int_0^\infty e^{-pt} [e^{-at} f(t)] dt \\
&= \text{Lap } e^{-at} f(t).
\end{aligned} \tag{35}$$

This proves that *multiplying the function by e^{-at} corresponds to substituting $p + a$ for p in its Laplace transform.*

As an application of this substitution property, we may deduce from Eq. (32) that since

$$\frac{1}{p^2} = \text{Lap } t, \qquad \frac{1}{(p + a)^2} = \text{Lap } te^{-at}. \tag{36}$$

Let b be any positive number. Then if $f(t) = 0$ for $t < 0$, and the graph of $f(t)$ is translated or shifted b units to the right, the translated graph represents the function defined by

$$g(t) = 0 \quad \text{for } t < b, \qquad g(t) = f(t - b) \quad \text{for } t > b. \tag{37}$$

For example, Fig. 50 represents $f(t) = 0$ for $t < 0$ and $f(t) = t$

$f(t)=0,$
$t < 0$

$f(t)=t,$
$t > 0$
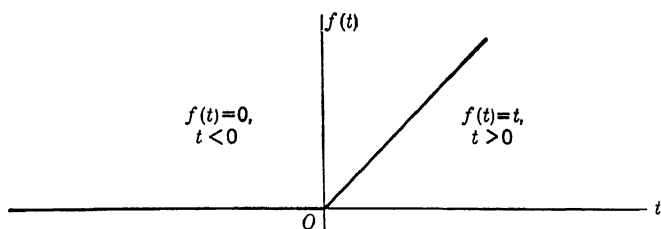
Fɪɢ. 50.   Graph of a particular $f(t)$.

$g(t)=0,$
$t < b$
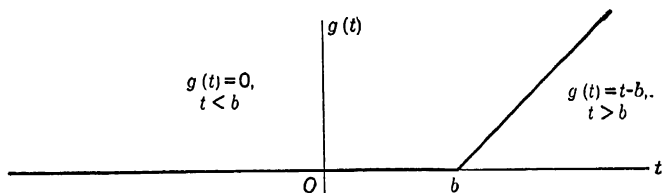
$g(t)=t-b,.$
$t > b$

Fɪɢ. 51.   Graph of $g(t) = f(t - b)$ is obtained by translating graph of $f(t)$ b units to the right.

for $t > 0$. And Fig. 51 represents $g(t) = 0$ for $t < b$, and $g(t) = t - b$ for $t > b$.

For the $g(t)$ of Eq. (37), we may use Eq. (1) to find

$$G(p) = \text{Lap } g(t).$$

We have

$$G(p) = \int_0^\infty e^{-pt}g(t)dt = \int_0^b e^{-pt}g(t)dt + \int_b^\infty e^{-pt}g(t)dt$$
$$= \int_0^b e^{-pt}0\, dt + \int_b^\infty e^{-pt}f(t-b)dt$$
$$= \int_b^\infty e^{-pu}f(u-b)du. \quad (38)$$

Here we have used the values of $g(t)$ for the intervals $0,b$ and b,∞ as given in Eq. (37), noted that the integral from 0 to b is zero and we have replaced the variable of integration t by u.

Now make the substitution $u = t + b$ which implies that

$$du = dt, \qquad t = u - b \tag{39}$$

and $t = 0$ when $u = b$, $t = \infty$ when $u = \infty$. Thus from Eq. (38),

$$G(p) = \int_b^\infty e^{-pu}f(u-b)du = \int_0^\infty e^{-pt-pb}f(t)dt$$
$$= e^{-pb}\int_0^\infty e^{-pt}f(t)dt = e^{-pb}F(p). \tag{40}$$

This proves that for a function 0 for $t < 0$, *translating the function b units to the right corresponds to multiplying its transform by* e^{-bp}. More specifically, if $g(t)$ is the translated function given by Eq. (37), then Lap $g(t) = e^{-bp}$ Lap $f(t)$.

As an illustration of this translation property, we may deduce from Eq. (32) that for the $g(t)$ of Fig. 51, the Laplace transform is

$$e^{-bp} \text{ Lap } t = \frac{1}{p^2} e^{-bp}. \tag{41}$$

EXERCISE XXVII

Verify the following Laplace transforms by setting up and evaluating the integral of the defining equation, Eq. (1), in each case.

1. Lap $4 = \dfrac{4}{p}$.

2. Lap $4e^{-2t} = \dfrac{4}{p+2}$.

3. Lap $4t = \dfrac{4}{p^2}$.

4. Lap $(2 - 2e^{-2t}) = \dfrac{4}{p(p+2)}$.

5. Lap $3te^{-2t} = \dfrac{3}{(p+2)^2}$.

6. Lap $(3 - 6t)e^{-2t} = \dfrac{3p}{(p+2)^2}$.

Use Eq. (8) to deduce the result of

7. Prob. 4 from Probs. 1 and 2.
8. Prob. 6 from Probs. 2 and 4.

Use the methods or results of Sec. 46 to verify that

9. Lap $2 \sinh 3t = \dfrac{6}{p^2 - 9}$.   **10.** Lap $2 \cosh 3t = \dfrac{2p}{p^2 - 9}$.

11. Lap $(\cosh 3t - 1) = \dfrac{9}{p(p^2 - 9)}$.   **12.** Lap $3 \sin 2t = \dfrac{6}{p^2 + 4}$.

13. Lap $3 \cos 2t = \dfrac{3p}{p^2 + 4}$.   **14.** Lap $(\cos 2t - 1) = \dfrac{-4}{p(p^2 + 4)}$.

15. By evaluating each of its terms, check Eq. (22) when $f(t) = e^{2t}$.

16. Use Probs. 9 and 10 to check Eq. (19) when
$$f(t) = 2 \cosh 3t.$$

17. Use Probs. 12 and 13 to check Eq. (19) when
$$f(t) = 3 \cos 2t.$$

18. Check Eq. (26) when $f(t) = 2e^{-t}$ and $t_0 = 1$.

Show that Eq. (29) may be used to deduce the result of

19. Prob. 1 from that of Prob. 3.
20. Prob. 2 from that of Prob. 4.
21. Prob. 6 from that of Prob. 5.

Show that Eqs. (31) and (6) may be used to deduce the result of

22. Prob. 9 from that of Prob. 10.
23. Prob. 11 from that of Prob. 9.
24. Prob. 12 from that of Prob. 13.
25. Prob. 14 from that of Prob. 12.

26. Show that Lap $(1 - 2t) = \dfrac{p - 2}{p^2}$.

27. Show that the Laplace transform of the polynomial $a + bt + ct^2 + dt^3$ is $\dfrac{ap^3 + bp^2 + 2cp + 6d}{p^4}$.

From the substitution property, Eq. (35), and the result of

28. Prob. 12 deduce that Lap $3e^{-t} \sin 2t = \dfrac{6}{(p+1)^2 + 4}$.

29. Prob. 13 deduce that Lap $3e^{-t} \cos 2t = \dfrac{3p+3}{(p+1)^2 + 4}$.

30. Prob. 26 deduce that Lap $(1 - 2t)e^{-2t} = \dfrac{p}{(p+2)^2}$.

31. Use Eqs. (17) and (6) to check Prob. 28.
32. Use Eqs. (16) and (6) to check Prob. 29.
33. Use Prob. 5 and Eqs. (29) and (6) to check Prob. 30.

From the translation property, Eqs. (37) and (40), show that

34. If $f(t) = 0$ for $t < 3$ and $f(t) = 5$ for $t > 3$, then
Lap $f(t) = \dfrac{5}{p} e^{-3p}$.

35. If $f(t) = 0$ for $t < 2\pi$ and $f(t) = \sin t$ for $t > 2\pi$, then
Lap $f(t) = \dfrac{e^{-2\pi p}}{p^2 + 1}$.

36. If $f(t) = 0$ for $t < \pi$ and $f(t) = \sin t$ for $t > \pi$, then
Lap $f(t) = \dfrac{-e^{-\pi p}}{p^2 + 1}$.

37. If $f(t) = 0$ for $t < \pi/2$ and $f(t) = \sin t$ for $t > \pi/2$, then
Lap $f(t) = \dfrac{pe^{-\pi p/2}}{p^2 + 1}$.

51. Differential Equations

Many problems involving differential equations may be solved by means of Laplace transforms. To illustrate the method we consider the following problem:

Find the solution of the differential equation

$$\frac{dy}{dt} = 4 \qquad \text{for } t > 0, \tag{42}$$

which has $y = 5$ when $t = 0$.

If $y = f(t)$, $f(0+) = 5$. Hence if we call the transform of our

function $F(p)$, by Eq. (19) we have

$$\text{Lap } \frac{dy}{dt} = \text{Lap } f'(t) = -f(0+) + p \text{ Lap } f(t)$$
$$= -5 + pF(p). \tag{43}$$

The transform of the right member of Eq. (42) is found from Eqs. (6) and (4) to be

$$\text{Lap } 4 = \text{Lap } 4 \cdot 1 = 4 \text{ Lap } 1 = 4\frac{1}{p} = \frac{4}{p}. \tag{44}$$

If we take the Laplace transforms of both sides of Eq. (42), in view of Eqs. (43) and (44) we find that

$$-5 + pF(p) = \frac{4}{p}. \tag{45}$$

This may be solved for $F(p)$ to give

$$F(p) = \frac{4}{p^2} + \frac{5}{p}. \tag{46}$$

By Eq. (32), $\text{Lap}^{-1}\frac{1}{p^2} = t$. And by Eq. (4), $\text{Lap}^{-1}\frac{1}{p} = 1$. Hence by Eq. (8),

$$f(t) = \text{Lap}^{-1}\left(\frac{4}{p^2} + \frac{5}{p}\right) = 4 \text{ Lap}^{-1}\frac{1}{p^2} + 5 \text{ Lap}^{-1}\frac{1}{p}$$
$$= 4t + 5. \tag{47}$$

This is the solution of our problem. It is a solution because our steps are reversible. And it is the only solution because for $t > 0$, it consists of one regular piece, and inside a regular piece the inverse Laplace transform is uniquely determined.

We have introduced $f(t)$ and $F(p)$ to conform to our earlier notation. Since the variable here was y, it would be more natural to use Y for $Y(p)$, the transform of $y(t)$. With this notation we would write in place of Eqs. (45), (46), and (47)

$$-5 + pY = \frac{4}{p}, \qquad Y = \frac{4}{p^2} + \frac{5}{p}, \qquad y = 4t + 5. \tag{48}$$

Let us consider the Laplace transform method of solving a single linear differential equation with constant coefficients. If the right member is a linear combination of the simple functions whose transforms were found in Secs. 47 and 49, the transform of that member may be found by applying these results and combining them by using Eqs. (6), (7), and (8).   The transform of the left member may be found by using these same equations to combine the results of Sec. 48, or for a dead system those of Sec. 49. The transformed equation may contain polynomial or rational functions of p, but will involve Lap $f(t) = F(p)$, or Lap $y = Y$ to the first power only.   Hence it is easy to solve for Y as a function of p.   If the function of t which has this as its transform can be found, it will be the solution of our given equation and initial values.

The final step of the above procedure, as well as the evaluation of the transform of the right member, is facilitated by consulting a table of some common functions, their transforms, and the effect on the transforms of certain operations on functions.   The table given in the next section, and repeated at the end of the book, is adequate for all the problems of this chapter.

52. Table of Transforms

For easy reference, we collect several of our results in tabular form.   Most of these were established in Secs. 47 to 50.   For the derivations of the rest, see Probs. 38, 39, 40, and 46 of Exercise XXVIII.

We note that items 1 to 13 involve general functions and express a correspondence between operations on the function and other operations on the transform.   Specifically, to find the transforms of derivatives in the general case we use items 2, 3, and 4, while for derivatives in problems on systems dead initially we use items 7, 8, and 9.   And to transform integrals we use items 5, 6, and 10. Item 1 expresses the linearity property by which the problem of finding the transform, or inverse transform, of a linear combination of terms is reduced to the corresponding problem for the individual terms.   Item 11, the substitution property, item 12,

TABLE OF LAPLACE TRANSFORMS

	Function $f(t) = \text{Lap}^{-1} F(p)$	Transform $\text{Lap } f(t) = F(p)$
1	$c_1 f(t) + c_2 g(t)$	$c_1 F(p) + c_2 G(p)$
2	$f'(t)$	$-f(0+) + pF(p)$
3	$f''(t)$	$-f'(0+) - pf(0+) + p^2 F(p)$
4	$f^{(n)}(t)$	$-f^{(n-1)}(0+) - pf^{(n-2)}(0+) - \cdots - p^{n-1}f(0+) + p^n F(p)$
5	$\int_{t_0}^t f(u)du$	$\frac{1}{p} F(p) + \frac{1}{p}\int_{t_0}^0 f(u)du$
6	$\int_0^t f(u)du$	$\frac{1}{p} F(p)$
7	$f'(t)$, when $f(0+) = 0$	$pF(p)$
8	$f''(t)$, when $f(0+) = 0, f'(0+) = 0$	$p^2 F(p)$
9	$f^{(n)}(t)$ when $f(0+) = \cdots = f^{(n-1)}(0+) = 0$	$p^n F(p)$
10	$q = \int_{t_0}^t i\, dt$	$\frac{I + q_0}{p}$
11	$e^{-at}f(t)$	$F(p + a)$
12	$g(t) = 0$ for $t < b$ $g(t) = f(t - b)$ for $t > b$	$G(p) = e^{-bp}F(p)$
13	$h(t) = \int_0^t f(u)g(t - u)du$ $= \int_0^t f(t - u)g(u)du$	$H(p) = F(p)G(p)$
14	1	$\frac{1}{p}$

TABLE OF LAPLACE TRANSFORMS.—(*Continued*)

	Function $f(t) = \mathrm{Lap}^{-1}\, F(p)$	Transform $\mathrm{Lap}\, f(t) = F(p)$
15	e^{-at}	$\dfrac{1}{p + a}$
16	t	$\dfrac{1}{p^2}$
17	$\dfrac{t^2}{2}$	$\dfrac{1}{p^3}$
18	$\dfrac{t^n}{n!}$	$\dfrac{1}{p^{n+1}}$
19	$\sin kt$	$\dfrac{k}{p^2 + k^2}$
20	$\cos kt$	$\dfrac{p}{p^2 + k^2}$
21	$\sinh kt$	$\dfrac{k}{p^2 - k^2}$
22	$\cosh kt$	$\dfrac{p}{p^2 - k^2}$
23	te^{-at}	$\dfrac{1}{(p + a)^2}$
24	$\dfrac{t^n}{n!}\, e^{-at}$	$\dfrac{1}{(p + a)^{n+1}}$
25	$e^{-at} \sin kt$	$\dfrac{k}{(p + a)^2 + k^2}$
26	$e^{-at} \cos kt$	$\dfrac{p + a}{(p + a)^2 + k^2}$

the translation property, and item 13, the convolution property, are all used for finding either direct or inverse transforms for terms of the appropriate form.

The last portion of the table, items 14 to 26, gives the transforms of particular functions. We may either find a given function of t in the first column and obtain its transform from the second column, or find a given function of p in the second column and obtain its inverse transform from the first column.

The use of the Table of Laplace Transforms first to introduce transforms of given functions of t and later on to find a desired solution from its transform function of p is roughly analogous to the use of a table of logarithms to facilitate numerical computations.

EXERCISE XXVIII

Find the Laplace transform of each of the following given functions of t by using the Table of Laplace Transforms.

1. 5. **2.** $5t$. **3.** $(4 - 2t)$.

4. $5e^{-t}$. **5.** $2te^{-3t}$. **6.** $(2 - 6t)e^{-3t}$.

7. $4 \sinh 5t$. **8.** $4 \cosh 5t$. **9.** $\cosh 4t - 1$.

10. $5 \sin 3t$. **11.** $5 \cos 3t$. **12.** $\cos 5t - 1$.

13. $t^3 - 2t^4$. **14.** $e^{-2t} \sin t$. **15.** $e^{-3t} \cos 2t$.

16. $f(t) = 0$ for $t < 4$ and $f(t) = 6$ for $t > 4$.
17. $f(t) = 0$ for $t < 4$ and $f(t) = 2t - 8$ for $t > 4$.
18. $f(t) = 0$ for $t < 4$ and $f(t) = 2t$ for $t > 4$.
19. $f(t) = 0$ for $t < 2$ and $f(t) = e^{2-t}$ for $t > 2$.
20. $f(t) = 0$ for $t < 2$ and $f(t) = e^{-t}$ for $t > 2$.
21. $f(t) = 0$ for $t < \pi$ and $f(t) = \sin t$ for $t > \pi$.

For each of the following given functions of p, use the table to find the inverse Laplace transform, that is, the function of t which has the given function as its Laplace transform.

22. $\dfrac{3}{p}$. **23.** $\dfrac{7}{p^2}$. **24.** $\dfrac{2 + 6p}{p^3}$.

25. $\dfrac{4}{p+5}.$

26. $\dfrac{2}{(p+3)^2}.$

27. $\dfrac{6}{p^2+3p} = \dfrac{2}{p} - \dfrac{2}{p+3}.$

28. $\dfrac{4}{p^2+4}.$

29. $\dfrac{4p}{p^2+4}.$

30. $\dfrac{2p}{(p+3)^2} = \dfrac{2}{p+3} - \dfrac{6}{(p+3)^2}$

31. $\dfrac{4}{p^2-4}.$

32. $\dfrac{4p}{p^2-4}.$

33. $\dfrac{2}{p(p^2+1)} = \dfrac{2}{p} - \dfrac{2p}{p^2+1}.$

34. $\dfrac{e^{-\pi p}}{p^2+1}.$

35. $\dfrac{6}{(p+2)^2+9}$

36. $\dfrac{6p}{(p+2)^2+9}$

37. Deduce item 4 of the table for $n = 3$ from items 2 and 3, or Eqs. (21) and (22).

38. Prove item 4 for n any positive integer by mathematical induction.

39. Prove item 9 for n any positive integer by mathematical induction.

40. Deduce the properties of the convolution expressed in item 13 from the results of Probs. 15, 16, and 17 of Exercise XI by making suitable changes in the notation.

Use item 11 of the table to deduce

41. Item 15 from item 1.

42. Item 23 from item 16.

43. Item 24 from item 18.

44. Item 25 from item 19.

45. Item 26 from item 20.

46. Using item 6 and mathematical induction, prove item 18.

47. Show that if $g(t) = 1$, item 13 is equivalent to item 6.

Verify that item 13 gives a correct result when

48. $f(t) = t,$ $g(t) = t.$ **49.** $f(t) = e^{-2t},$ $g(t) = e^{-2t}.$

Verify the following statements, which lead up to the relations of Probs. 54 and 55 involving $pF(p) = p$ Lap $f(t)$

50. For any small positive number b, and large positive number B

$$p \int_b^B e^{-pt}\, dt = e^{-pb} - e^{-pB}.$$ Hence if $|g(t)| < M$ for

$$b < t < B,$$

$\left| p \int_b^B e^{-pt} g(t)\, dt \right| < M(e^{-pb} - e^{-pB})$ which approaches zero when $p \to +\infty$. It also approaches zero when $p \to 0$, even when $b = 0$.

51. If $|g(t)| < e^{at}$ for $t > B$,

$$\left| p \int_B^\infty e^{-pt} g(t)\, dt \right| < p \int_B^\infty e^{(a-p)t}\, dt$$

or for $p > a$, $\dfrac{p}{p-a} e^{(a-p)B}$ which approaches zero when $p \to +\infty$.

52. $p \int_0^\infty e^{-pt}\, dt = 1$. Hence $f(0+) = p \int_0^\infty e^{-pt} f(0+)\, dt$. Also since pe^{-pt} is positive, $p \int_0^b e^{-pt}\, dt < 1$, and

$$\left| p \int_0^b e^{-pt} [f(t) - f(0+)]\, dt \right| < m$$

if $|f(t) - f(0+)| < m$ for $0 < t < b$.

53. Also from the equations of Prob. 52,

$$f(+\infty) = p \int_0^\infty e^{-pt} f(+\infty)\, dt.$$

And $p \int_B^\infty e^{-pt} < 1$, and $\left| p \int_B^\infty e^{-pt} [f(t) - f(+\infty)]\, dt \right| < m$ if $|f(t) - f(+\infty)| < m$ for $t > B$.

54. If as t approaches zero through positive values, $f(t) \to f(0+)$, then $\lim\limits_{p \to \infty} pF(p) = f(0+)$. This is the *initial value* theorem.

Prove it by the following steps:

$pF(p) - f(0+) = p \int_0^\infty e^{-pt} [f(t) - f(0+)]\, dt$, by Prob. 52. Next write the integral from 0 to ∞ as the sum of three integrals, I from 0 to b, II from b to B, and III from B to ∞. With

$$g(t) = f(t) - f(0+),$$

by Prob. 50, II → 0 and by Prob. 51, III → 0 when $p = +\infty$. Hence $|II| < m$ and $|III| < m$ for $p > p_1$. And by Prob. 52, $|I| < m$ if b is so small that $|f(t) - f(0+)| < m$ for $0 < t < b$. Since we may choose m arbitrarily small, and then find a suitable b, it follows that $|pF(p) - f(0+)| < 3m$ for $p > p_1$. Hence $pF(p) - f(0+) → 0$, and $pF(p) → f(0+)$ when $p → \infty$.

55. If as t becomes positively infinite, $f(t) → f(\infty +)$, a finite limit, then $\lim\limits_{p→0} pF(p) = f(+\infty)$. This is the *final value theorem*.

Prove it by the following steps:

$pF(p) - f(+\infty) = p \int_0^\infty e^{-pt}[f(t) - f(+\infty)]dt$, by Prob. 53. Next write the integral from 0 to ∞ as the sum of two integrals, I from 0 to B, and II from B to ∞. With $g(t) = f(t) - f(\infty +)$, I → 0 by the last statement of Prob. 50 when $p → 0$. Hence $|I| < m$ for $p < p_1$. And by Prob. 53, $|II| < m$ if B is so large that $|f(t) - f(+\infty)| < m$ for $t > b$. Since we may choose m arbitrarily small, and then find a suitable B, it follows that

$$|pF(p) - f(+\infty)| < 2m$$

for $p < p_1$. Hence $pF(p) - f(+\infty) → 0$, and $pF(p) → f(+\infty)$ when $p → 0$.

56. Verify that for items 14, 15, 20, 22, and 26, $f(0+) = 1$, and $\lim\limits_{p→\infty} pF(p)$ also $= 1$, in accord with the initial value theorem of Prob. 54.

57. Verify that for items 16, 19, 21, 23 and 25, $f(0+) = 0$, $f(t) → 0$ like t, while $\lim\limits_{p→\infty} pF(p) → 0$, with $pF(p) → 0$ like $1/p$.

58. Verify that for items 18 and 24, $f(0+) → 0$, $f(t) → 0$ like t^n, while $\lim\limits_{p→\infty} pF(p) = 0$, with $pF(p) → 0$ like $1/p^n$.

59. Verify that $f(+\infty)$ and $\lim\limits_{p→0} pF(p)$ are each 1 for item 14, and each zero for item 15 with $a > 0$, items 16, 17, 18, and 23, 24, 25, and 26, with $a > 0$. This is in accord with the final value theorem of Prob. 55.

60. Prove that Lap $f(at) = \dfrac{1}{a} F\left(\dfrac{p}{a}\right)$, by making a change of variable $at = u$.

61. Show that $\text{Lap}^{-1} F(bp) = \dfrac{1}{b} f\left(\dfrac{t}{b}\right)$ by using Prob. 60 with $a = \dfrac{1}{b}$.

62. Assuming items 15, 23, and 24 given for $a = 1$, use Prob. 60 to deduce these items for any a.

63. Assuming items 19 and 20 given for $k = 1$, use Prob. 60 to deduce these items for any k.

64. In Prob. 55 we assumed that $\lim\limits_{t \to +\infty} f(t)$ was finite. Show that the final value theorem does not necessarily hold when $\lim\limits_{t \to +\infty} f(t) = +\infty$ by considering items 15 with $a > 0$ and items 21 and 22.

53. General Initial Conditions

The particular solution of a differential equation may be prescribed by fixing the initial values of the unknown function and certain of its derivatives, the disposable initial values for the equation or system which gave rise to it. These values are all zero for any system which was dead initially. In most of the following paragraphs we assume this condition to hold. This will enable us to use the simplified relations of Sec. 49, or items 6 to 9 of the table. For such systems an additional merit of the transform method is that we can obtain the solution without bothering to figure our separately how many initial values must be assigned as zero.

But in this section we consider some applications of the transform method to systems not dead originally but having general initial conditions. In such cases, when finding the transforms of derivatives, we must use Eqs. (19) and (22), or items 2, 3, and 4 of the Table of Laplace Transforms.

As an illustration, suppose that we wish to find the solution

of the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 13x = 0 \tag{49}$$

which satisfies the initial conditions

$$x = 3, \qquad \frac{dx}{dt} = 6 \qquad \text{for } t = 0. \tag{50}$$

We apply items 2 and 3 with x, X, dx/dt, d^2x/dt^2 in place of f, F, f', f'', respectively. And we set $f(0+) = x(0+) = 3$ and $f'(0+) = x(0+) = 6$ in accordance with Eq. (50). Thus we obtain

$$\text{Lap } \frac{dx}{dt} = -3 + pX, \tag{51}$$

$$\text{Lap } \frac{d^2x}{dt^2} = -6 - 3p + p^2X. \tag{52}$$

Hence for Eq. (49) the transformed equation is

$$-6 - 3p + p^2X + 4(-3 + pX) + 13X = 0.$$

or

$$(p^2 + 4p + 13)X = 3p + 18.$$

It follows that

$$X = \frac{3p + 18}{p^2 + 4p + 13}. \tag{53}$$

The denominator is $(p + 2)^2 + 9$. Hence we write the numerator as $3(p + 2) + 12$ and

$$X = 3\frac{p + 2}{(p + 2)^2 + 3^2} + 4\frac{3}{(p - 2)^2 + 3^2}. \tag{54}$$

Finally, from items 1, 26, and 25 of the table, we have as the solution

$$x = 3e^{-2t} \cos 3t + 4e^{-2t} \sin 3t. \tag{55}$$

As a second example, consider a condenser of capacity C initially carrying a charge q_0 and discharging through a resistance

R. Here the differential equation for the charge q is

$$R \frac{dq}{dt} + \frac{q}{C} = 0, \tag{56}$$

and $q = q_0$ when $t = 0$.

To find q, we may apply item 2 of the table with q, Q, dq/dt in place of f, F, f', respectively. And we set $f(0+) = q(0+) = q_0$. Thus

$$\text{Lap } \frac{dq}{dt} = -q_0 + pQ. \tag{57}$$

Hence for Eq. (56) the transformed equation is

$$R(-q_0 + pQ) + \frac{Q}{C} = 0, \quad \text{or} \quad (RCp + 1)Q = RCq_0.$$

It follows that

$$Q = \frac{RCq_0}{RCp + 1} = q_0 \frac{1}{p + \dfrac{1}{RC}}. \tag{58}$$

Hence from items 1 and 15 of the table we obtain the value of q as

$$q = q_0 \epsilon^{-t/RC}. \tag{59}$$

Since the current $i = dq/dt$, i is given by

$$i = -\frac{q_0}{RC} \epsilon^{-t/RC}. \tag{60}$$

We might have used the fact that $i = dq/dt$ to write

$$Ri + \frac{q}{C} = 0 \quad \text{or} \quad Ri + \frac{1}{C} \int_{t_0}^{t} i \, dt = 0 \tag{61}$$

in place of Eq. (56). With I as the transform of i, and the given initial value q_0 we could apply item 10 of the table to either of these and thus obtain as the transformed equation

$$RI + \frac{1}{C}\left(\frac{I + q_0}{p}\right), \quad \text{or} \quad (RCp + 1)I = -q_0. \tag{62}$$

It follows that

$$I = \frac{-q_0}{RCp + 1} = -\frac{q_0}{RC} \frac{1}{p + \dfrac{1}{RC}}. \tag{63}$$

Hence from items 1 and 15 of the table we obtain the value of i as

$$i = -\frac{q_0}{RC} \epsilon^{-t/RC}. \tag{64}$$

If the value of i alone is desired, this alternative procedure based on Eq. (61) is preferable to the method used to obtain Eq. (60).

54. Partial Fractions

For problems involving a forcing function, or nonzero right-hand member, the Laplace transform of the solution is usually too complicated to be found directly in our short table.   But it may often be decomposed into simpler parts each of a type found in the table by the method of partial fractions, which we shall now discuss.

As our first example, let us find the solution of the equation

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 1 \tag{65}$$

which satisfies the initial conditions $x = 0$ and $dx/dt = 0$ when $t = 0$.   As this system is initially dead, we may apply items 7 and 8 of the table with x, X, dx/dt, d^2x/dt^2 in place of j, F, j', f'' to obtain

$$\text{Lap}\frac{dx}{dt} = pX \qquad \text{and} \qquad \text{Lap}\frac{d^2x}{dt^2} = p^2X \tag{66}$$

And by item 14, Lap $1 = 1/p$.   Hence the result of taking the Laplace transform of each member of Eq. (65) is

$$p^2X - 3pX + 2X = \frac{1}{p}, \qquad \text{or} \qquad (p^2 - 3p + 2)X = \frac{1}{p}. \tag{67}$$

It follows that

$$X = \frac{1}{p(p^2 - 3p + 2)} = \frac{1}{p(p - 1)(p - 2)}. \tag{68}$$

The fraction just written is proper, that is, its denominator is of higher degree than the numerator. And the denominator is the product of distinct first-degree factors. Hence there exist constants A, B, and C such that

$$\frac{1}{p(p - 1)(p - 2)} = \frac{A}{p} + \frac{B}{p - 1} + \frac{C}{p - 2}. \tag{69}$$

The three constants A, B, C could be evaluated by clearing of fractions, equating coefficients of corresponding powers of p, and solving the set of three linear equations that result. But there are two simpler methods.

To derive these, consider the general relation

$$\frac{N(p)}{D(p)} = \frac{A_1}{p - r_1} + \frac{M(p)}{E_1(p)}. \tag{70}$$

Here A_1 is a constant, $N(p)$, $M(p)$, $D(p)$, $E_1(p)$ are all polynomials, and r_1 is a simple root of $D(p) = 0$, so that

$$D(p) = (p - r_1)E_1(p), \qquad D(r_1) = 0, \qquad E_1(r_1) \neq 0. \tag{71}$$

Let us multiply both sides of Eq. (70) by $p - r_1$, and then let $p \to r_1$. The first term on the right is A_1, and the other term contains the factor $p - r_1$, and thus approaches 0 when $p \to r_1$. Thus

$$A_1 = \lim_{p \to r_1} (p - r_1) \frac{N(p)}{D(p)} = N(r_1) \lim_{p \to r_1} \frac{p - r_1}{D(p)}. \tag{72}$$

If we use the first relation of Eq. (71), and cancel the factor $(p - r_1)$, we see that the last limit in Eq. (72) is $1/E_1(r_1)$. Hence

$$A_1 = \frac{N(r_1)}{E_1(r_1)} = \frac{N(p)}{E_1(p)}\bigg|_{p = r_1}. \tag{73}$$

This leads to the first rule: In Eq. (70), A_1 *may be found by deleting*

the factor $p - r_1$ from the denominator $D(p)$ and then evaluating the left member for $p = r_1$.

Or we may apply l'Hospital's rule to the last limit in Eq. (72) and thus obtain $1/D'(r_1)$.   Hence we have our second rule:

$$A_1 = \frac{N(r_1)}{D'(r_1)} = \frac{N(p)}{dD/dp}\bigg|_{p=r_1}. \tag{74}$$

On applying the first rule to Eq. (69) we find that

$$A = \frac{1}{(p-1)(p-2)}\bigg|_{p=0} = \frac{1}{2}, \qquad B = \frac{1}{p(p-2)}\bigg|_{p=1} = -1,$$

$$C = \frac{1}{p(p-1)}\bigg|_{p=2} = \frac{1}{2}. \tag{75}$$

To apply the second rule, Eq. (74), we note that

$$D(p) = p(p-1)(p-2) = p^3 - 3p^2 + 2p,$$

so that

$$D'(p) = \frac{dD}{dp} = 3p^2 - 6p + 2.$$

Hence the right member of Eq. (74) for this case becomes

$$\frac{N(p)}{D'(p)} = \frac{1}{3p^2 - 6p + 2} \tag{76}$$

Evaluating this for each of the roots 0, 1, 2 we find that

$$A = \frac{N(0)}{D'(0)} = \frac{1}{2} \qquad B = \frac{N(1)}{D'(1)} = -1, \qquad C = \frac{N(2)}{D'(2)} = \frac{1}{2}. \tag{77}$$

Equations (77) and (75) check, and either process shows that from Eqs. (68) and (69),

$$X = \frac{1}{2}\frac{1}{p} - \frac{1}{p-1} + \frac{1}{2}\frac{1}{p-2}. \tag{78}$$

Finally, from items 1, 14, and 15 of the table we find the solution

$$x = \tfrac{1}{2} - e^t + \tfrac{1}{2}e^{2t}. \tag{79}$$

Our two rules determine all the constants only when the factors

in the denominator of the expression to be expanded are linear and nonrepeated.   In case a factor is repeated k times we must assume k terms in the partial fraction expansion, corresponding to the successive powers of the repeated factor.   For example,

$$\frac{1}{(p-1)^2(p-2)} = \frac{A}{p-2} + \frac{B}{p-1} + \frac{C}{(p-1)^2}. \tag{80}$$

Here the first rule may be used to find A as

$$A = \frac{1}{(p-1)^2}\Big|_{p=2} = 1. \tag{81}$$

And on multiplying both sides of Eq. (80) by $(p-1)^2$ and letting $p \to 1$, we find C as

$$C = \frac{1}{p-2}\Big|_{p=1} = -1. \tag{82}$$

Since A and C are known, we may find B in this case by putting $A = 1, C = -1$, and p any particular value distinct from 1 and 2, for example, $p = 0$.   Using these values in Eq. (80), we find

$$-\tfrac{1}{2} = -\tfrac{1}{2} - B - 1 \qquad \text{so that} \qquad B = -1. \tag{83}$$

For two double factors or one triple factor there would be two constants not found from our modified rule.   In such a case we would use two particular values of p, to obtain two simultaneous equations for the two unknown constants.

The reader may easily verify that the relation

$$\frac{1}{(p-1)^2(p-2)} = \frac{1}{p-2} + \frac{-1}{p-1} + \frac{-1}{(p-1)^2} \tag{84}$$

which results from combining Eqs. (80), (81), (82), and (83) is a correct identity by adding the fractions on the right.

To illustrate the modifications of our methods necessary when quadratic factors arise, we shall find the solution of the equation

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 10 \tag{85}$$

which satisfies the initial conditions $x = 0$ and $dx/dt = 0$ when $t = 0$. From items 1 and 14, Lap $10 = 10/p$. Using this and Eq. (66) to take the Laplace transform of Eq. (85), we obtain

$$p^2 X + 2pX + 5X = \frac{10}{p}, \quad \text{or} \quad (p^2 + 2p + 5)X = \frac{10}{p}. \quad (86)$$

It follows that

$$X = \frac{10}{p(p^2 + 2p + 5)} \quad (87)$$

In this case constants A, B, and C exist such that

$$\frac{10}{p(p^2 + 2p + 5)} = \frac{A}{p} + \frac{Bp + C}{p^2 + 2p + 5}. \quad (88)$$

Note that for a quadratic factor in the denominator of a simple fraction, we must assume a first-degree numerator.

Since zero is a simple root, the first rule may be used to find A as

$$A = \frac{10}{p^2 + 2p + 5}\bigg|_{p=0} = 2. \quad (89)$$

To evaluate B and C, we multiply Eq. (88) by $p^2 + 2p + 5$ and let $p \to -1 + 2i$, one root of $p^2 + 2p + 5 = 0$. This leads to

$$\frac{10}{-1 + 2i} = B(-1 + 2i) + C. \quad (90)$$

By the procedure of Eq. (6) of Sec. 1, we have

$$\frac{10}{-1 + 2i} = \frac{10}{-1 + 2i}\frac{-1 - 2i}{-1 - 2i} = \frac{10}{5}(-1 - 2i) = -2 - 4i$$

so that

$$(91)$$

$$-2 - 4i = (-B + C) + 2Bi. \quad (92)$$

Equating real and imaginary parts separately, we have

$$-2 = -B + C, \quad -4 = 2B, \quad \text{so that} \quad B = -2$$
$$\text{and } C = B - 2 = -4. \quad (93)$$

From Eqs. (87) and (88) and the values given in Eqs. (89) and (93)

we find

$$X = \frac{2}{p} + \frac{-2p - 4}{p^2 + 2p + 5}. \tag{94}$$

When, as here, there is only one quadratic factor, the numerator may be found by subtraction. Thus, after finding $A = 2$ in Eq. (89), we might have deduced from Eq. (88) that

$$\begin{aligned}
\frac{Bp + C}{p^2 + 2p + 5} &= \frac{10}{p(p^2 + 2p + 5)} - \frac{2}{p} \\
&= \frac{-2p^2 - 4p}{p(p^2 + 2p + 5)} \\
&= \frac{-2p - 4}{p^2 + 2p + 5}. \tag{95}
\end{aligned}$$

This shows that $B = -2$ and $C = -4$ as in Eq. (93) and therefore gives Eq. (94).

To take the inverse transform, we rewrite Eq. (94) in the form

$$X = 2\frac{1}{p} - 2\frac{p + 1}{(p + 1)^2 + 2^2} - \frac{2}{(p + 1)^2 + 2^2}. \tag{96}$$

We now apply items 1, 14, 26, and 25 of the Table of Laplace Transforms to deduce that

$$x = 2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t. \tag{97}$$

This is the desired solution of Eq. (85).

It is possible to replace quadratic factors by products of first-degree factors involving complex roots. As this is sometimes desirable when there are several complex roots in the denominator, we shall illustrate the method as applied to Eq. (87). Since the roots of $p^2 + 2p + 5 = 0$ are $-1 \pm 2i$, we have

$$\begin{aligned}
\frac{10}{p(p^2 + 2p + 5)} &= \frac{10}{p(p + 1 + 2i)(p + 1 - 2i)} \\
&= \frac{A}{p} + \frac{D}{p + 1 - 2i} + \frac{\bar{D}}{p + 1 + 2i} \\
&= \frac{A}{p} + 2\,\text{Re}\,\frac{D}{p + 1 - 2i}. \tag{98}
\end{aligned}$$

Since the left side is real, the two fractions with complex factors will be conjugate. Thus D and $\bar{D}$ will be conjugate complex numbers. Once the complex roots are known, it is only necessary to use one of each pair, and to take twice the real part of the corresponding fraction as in the last expression of Eq. (98).

We find $A = 2$ as in Eq. (89). And we may also use the first rule to find D, since the complex factors are not repeated. Thus

$$D = \frac{10}{p(p + 1 + 2i)}\bigg|_{p = -1+2i} = \frac{10}{(-1 + 2i)4i}$$
$$= \frac{1}{2i}(-1 - 2i). \qquad (99)$$

The last reduction is similar to Eq. (91). We have left the i in the denominator, because for any complex number $z = a + bi$,

$$\operatorname{Re}\frac{z}{i} = \operatorname{Re}(b - ai) = b = \operatorname{Im} z, \quad \text{and} \quad 2\operatorname{Re}\frac{z}{2i} = \operatorname{Im} z.$$
$$(100)$$

From Eqs. (87), (98), (89), (99), and (100) we may conclude that

$$X = 2\frac{1}{p} + \operatorname{Im}(-1 - 2i)\frac{1}{p + 1 - 2i}. \qquad (101)$$

We now apply items 1, 14, and 15 of the Table of Laplace Transforms to deduce that

$$x = 2 + \operatorname{Im}(-1 - 2i)e^{-(1-2i)t}. \qquad (102)$$

But by Sec. 2,
$$e^{-(1-2i)t} = e^{-t}e^{2it} = e^{-t}(\cos 2t + i\sin 2t),$$
and
$$(-1 - 2i)(\cos 2t + i\sin 2t) = \cdots + i(-2\cos 2t - \sin 2t),$$
so that
$$\operatorname{Im}(-1 - 2i)e^{-(1-2i)t} = e^{-t}(-2\cos 2t - \sin 2t). \qquad (103)$$

Equations (102) and (103) show that

$$x = 2 - 2e^{-t}\cos 2t - e^{-t}\sin 2t. \qquad (104)$$

This checks the result found in Eq. (97).

By the procedure of Sec. 6, for the factor in Eq. (102), we have
$-1 - 2i = re^{i\theta}$, where $r = \sqrt{5} = 2.2361$ and

$$\theta = \tan^{-1} \frac{-2}{-1} = -2.0345.$$

This shows that

$$(-1 - 2i)e^{-(1-2i)t} = re^{-t}e^{i(2t-\theta)}. \qquad (105)$$

We may combine these facts with Eq. (102) to deduce that

$$x = 2 + 2.2361e^{-t} \sin (2t - 2.0345). \qquad (106)$$

This alternative form is often preferable to that of Eq. (104).

As a further example, let us find the solution of the equation

$$\frac{d^2x}{dt^2} + a^2x = \sin kt, \qquad k \neq a \qquad (107)$$

which satisfies the initial conditions $x = 0$ and $dx/dt = 0$ when
$t = 0$. Using item 19 and Eq. (66) to take the Laplace transform
we obtain

$$p^2X + a^2X = \frac{k}{p^2 + k^2}, \qquad \text{or} \qquad (p^2 + a^2)X = \frac{k}{p^2 + k^2}. \qquad (108)$$

It follows that

$$X = \frac{k}{(p^2 + a^2)(p^2 + k^2)}. \qquad (109)$$

We first treat these real quadratic factors as in Eq. (88) and write

$$\frac{k}{(p^2 + a^2)(p^2 + k^2)} = \frac{Ap + B}{p^2 + a^2} + \frac{Cp + D}{p^2 + k^2} \qquad (110)$$

To evaluate A and B, multiply Eq. (110) by $p^2 + a^2$ and let
$p = ai$, one root of $p^2 + a^2 = 0$. Thus

$$\frac{k}{-a^2 + k^2} = Aai + B. \qquad (111)$$

Equating real and imaginary parts separately gives

$$B = \frac{k}{k^2 - a^2}, \qquad A = 0. \qquad (112)$$

To evaluate C and D, multiply Eq. (110) by $p^2 + k^2$ and let $p = ki$, one root of $p^2 + k^2 = 0$. Thus

$$\frac{k}{-k^2 + a^2} = Cki + D. \tag{113}$$

Equating real and imaginary parts separately gives

$$D = \frac{k}{a^2 - k^2}, \qquad C = 0. \tag{114}$$

From Eqs. (109) and (110) and the values given in Eqs. (112) and (114) we find

$$
\begin{aligned}
X &= \frac{k}{k^2 - a^2}\left(\frac{1}{p^2 + a^2} - \frac{1}{p^2 + k^2}\right) \\
&= \frac{k}{k^2 - a^2}\left(\frac{1}{a}\frac{a}{p^2 + a^2} - \frac{1}{k}\frac{k}{p^2 + k^2}\right).
\end{aligned} \tag{115}
$$

We now apply 1 and 19 of the Table of Laplace Transforms to deduce the solution

$$x = \frac{1}{a(k^2 - a^2)}\,(k \sin at - a \sin kt). \tag{116}$$

To treat Eq. (109) by the use of complex first-degree factors we note that the roots of the denominator are $\pm ai, \pm ki$. Hence, as in Eq. (98), we write here

$$
\begin{aligned}
X &= \frac{k}{(p^2 + a^2)(p^2 + k^2)} = \frac{k}{(p + ai)(p - ai)(p + ki)(p - ki)} \\
&= 2\,\mathrm{Re}\,\frac{E}{p - ai} + 2\,\mathrm{Re}\,\frac{F}{p - ki}.
\end{aligned} \tag{117}
$$

By applying the first rule in each case we find that

$$
\begin{aligned}
E &= \frac{k}{(p + ai)(p^2 + k^2)}\bigg|_{p=ai} = \frac{k}{2ai(k^2 - a^2)}, \\
F &= \frac{k}{(p^2 + a^2)(p + ki)}\bigg|_{p=ki} = \frac{1}{2i(a^2 - k^2)}.
\end{aligned} \tag{118}
$$

From Eqs. (117), (118), and (100) we may conclude that

$$X = \operatorname{Im} \frac{k}{a(k^2 - a^2)} \frac{1}{p - ai} + \operatorname{Im} \frac{1}{a^2 - k^2} \frac{1}{p - ki}. \quad (119)$$

We now apply items 1 and 15 of the Table of Laplace transforms to deduce that

$$x = \operatorname{Im} \frac{1}{a(k^2 - a^2)} (ke^{ait} - ae^{kit})$$

$$= \frac{1}{a(k^2 - a^2)} (k \sin at - a \sin kt). \quad (120)$$

This is in accord with the solution found in Eq. (116).

EXERCISE XXIX

1. Find the solution of $\dfrac{d^2x}{dt^2} + 4x = 0$ which has $x = 1$ and $\dfrac{dx}{dt} = 6$ when $t = 0$.

2. Find the solution of $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 10x = 0$ which has $x = 1$ and $\dfrac{dx}{dt} = 1$ when $t = 0$.

3. A condenser is discharged through an inductance, so that $L\dfrac{d^2q}{dt^2} + \dfrac{q}{c} = 0$. Find q and hence $i = \dfrac{dq}{dt}$, if $q = q_0$ and $i = 0$ when $t = 0$.

4. Check Prob. 3 by finding i directly as the appropriate solution of the equation $L\dfrac{di}{dt} + \dfrac{1}{C}\displaystyle\int_{t_0}^{t} i\, dt = 0$.

5. Find the solution of $\dfrac{d^2x}{dt^2} - 9x = 0$ which has $x = 5$ and $\dfrac{dx}{dt} = 3$ when $t = 0$.

6. Find the solution of $\dfrac{dx}{dt} + ax = 0$ which has $x = A$ when $t = 0$.

For each of the following equations, find the solution which has $x = A$ and $\dfrac{dx}{dt} = B$ when $t = 0$.   Assume that $b \neq 0$.

7. $\dfrac{d^2x}{dt^2} + b^2x = 0.$
8. $\dfrac{d^2x}{dt^2} + 2a\dfrac{dx}{dt} + (a^2 + b^2)x = 0.$

9. $\dfrac{d^2x}{dt^2} - b^2x = 0.$
10. $\dfrac{d^2x}{dt^2} + 2a\dfrac{dx}{dt} + a^2x = 0.$

For each of the following functions of p, find the decomposition into partial fractions of simple type.

11. $\dfrac{2p + 3}{p^2 - 5p + 6}.$
12. $\dfrac{p + 2}{(p - 1)(p - 3)(p - 4)}.$

13. $\dfrac{ap + bk}{p^2 - k^2}.$
14. $\dfrac{p - 3}{p^3 + 3p^2 + 2p}.$

15. $\dfrac{p^2 + p - 4}{p^3 - p^2}.$
16. $\dfrac{cp + d}{(p - a)(p - b)},\ a \neq b.$

17. $\dfrac{cp + d}{(p - a)^2}.$
18. $\dfrac{p}{(p^2 + 1)(p - 1)}.$

19. $\dfrac{4}{p^4 + p^2}.$
20. $\dfrac{25p^3}{(p^2 + 4)(p - 1)^2}.$

For each of the following equations, find the solution which has $x = 0$ and $\dfrac{dx}{dt} = 0$ when $t = 0$:

21. $\dfrac{d^2x}{dt^2} + 4x = 2.$
22. $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 2x = e^{-t}.$

23. $\dfrac{d^2x}{dt^2} - x = \cosh 2t.$
24. $\dfrac{d^2x}{dt^2} + 4\dfrac{dx}{dt} + 4x = e^{-2t}.$

25. $\dfrac{d^2x}{dt^2} + x = \sin 2t.$
26. $\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 4x = e^t.$

27. $\dfrac{d^2x}{dt^2} + x = \sin t.$ HINT: Find the complex constants in

$$\frac{1}{(p^2 + 1)^2} = \frac{A}{(p - i)^2} + \frac{\bar{A}}{(p + i)^2} + \frac{B}{p - i} + \frac{\bar{B}}{p + i}$$
$$= -\frac{1}{2}\operatorname{Re}\frac{1}{(p - i)^2} + \frac{1}{2}\operatorname{Im}\frac{1}{p - i}.$$

Find the solution of the differential equation $\dfrac{dx}{dt} + x = 3$ which has

28. $x = 0$ at $t = 0$.   **29.** $x = 6$ at $t = 0$.

30. By using item 13 with $g(t) = e^{-at}$, show that the solution of the equation $\dfrac{dx}{dt} + ax = f(t)$ which has $x = A$ when $t = 0$ is given by

$$x = e^{-at}\left[\int_0^t e^{au}f(u)du + A\right].$$

31. By using the partial fraction decomposition of $\dfrac{1}{p^2 - a^2}$ and item 13, show that the solution of $\dfrac{d^2x}{dt^2} - a^2x = f(t)$ which has $x = 0$ and $\dfrac{dx}{dt} = 0$ when $t = 0$ is given by

$$x = \frac{1}{2a}\,e^{at}\int_0^t e^{-au}f(u)du - \frac{1}{2a}\,e^{-at}\int_0^t e^{au}f(u)du.$$

32. Let $Q(p) = a_3p^3 + a_2p^2 + a_1p + a_0$ be a third-degree polynominal whose roots are r_1, r_2, r_3 so that

$$Q(p) = a_3(p - r_1)(p - r_2)(p - r_3).$$

If M is a constant, and the four numbers m, r_1, r_2, r_3 are all different, use rule II, Eq. (74), to verify that

$$\frac{M}{(p - m)Q(p)} = \frac{M}{Q(m)}\frac{1}{p - m} - \sum_{k=1}^{3}\frac{M}{(m - r_k)Q'(r_k)}\frac{1}{p - r_k}.$$

33. Use the notation, assumptions, and result of Prob. 32 to show that the solution of the differential equation

$$a_3\frac{d^3x}{dt^3} + a_2\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 = Me^{mt}$$

which has $x = 0$, $dx/dt = 0$, and $d^2x/dt^2 = 0$ when $t = 0$ is

$$x = \frac{M}{Q(m)} e^{mt} - \sum_{k=1}^{3} \frac{M}{(m - r_k)Q'(r_k)} e^{r_k t}.$$

This is a special case of the *Heaviside expansion*. A similar formula for an nth-order equation, with the summation running from 1 to n gives the solution whose derivatives vanish up to those of the $(n - 1)$st order, that is, for a system initially dead.

Use the Heaviside expansion of Prob. 33,

34. With $n = 2$ to check Prob. 22.
35. With $n = 1$ to check Prob. 28.

36. Let $Q(p) = a_3 p^3 + a_2 p^2 + a_1 p + a_0$ have three distinct roots r_1, r_2, r_3. Use rule II, Eq. (74), to verify that

$$\frac{1}{Q(p)} = \sum_{k=1}^{3} \frac{1}{Q'(r_k)} \frac{1}{p - r_k}.$$

37. Use item 13 with $g(t) = e^{r_k t}$ and the result of Prob. 36 to show that for a system initially dead the solution of the equation

$$a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 = f(t)$$

is

$$x = \sum_{k=1}^{3} \frac{1}{Q'(r_k)} e^{r_k t} \int_0^t e^{-r_k u} f(u) du.$$

38. Provided that all the roots are distinct, the result of Prob. 37 holds when 3 is replaced by n. Use this fact with $n = 2$ to check Prob. 31.

55. Series Circuits

We shall next apply the Laplace transform method to problems on electric networks and mechanical systems. We begin with the simple series circuits described in Sec. 11..

As a first example, let it be required to find the current $i(t)$ in the circuit of Fig. 52, assumed dead at $t = 0$. From Eq. (132) of Sec. 11 with $L = 2$, $R = 10$, $e = 15$, we have as the differential equation

$$2\frac{di}{dt} + 10i = 15. \tag{121}$$

As this system is initially dead, we may apply item 7 of the Table of Laplace Transforms with i, I, di/dt in place of f, F, f' to obtain
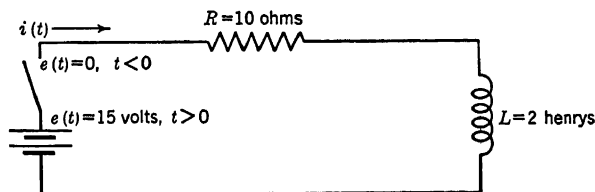
FIG. 52.

Lap $di/dt = pi$. And by items 1 and 14, Lap $15 = 15/p$. Hence the transformed equation is

$$2pI + 10I = \frac{15}{p}, \qquad \text{or} \qquad (2p + 10)I = \frac{15}{p}. \tag{122}$$

It follows that

$$I = \frac{15}{2}\frac{1}{p(p+5)} = \frac{3}{2}\left(\frac{1}{p} - \frac{1}{p+5}\right), \tag{123}$$

where the simple fractions are found as in Sec. 54.

From the last expression for I, using items 1, 14, and 15, we find

$$i = \frac{3}{2}(1 - \epsilon^{-5t}). \tag{124}$$

This is the resulting current. We note that $i = 0$ when $t = 0$, as it should for the solution of a first-order equation initially dead. And for $t = +\infty$, $i = \frac{3}{2}$, the current which would result if the inductance were short-circuited.

In practice it is not necessary to write down the differential equation (121), for the transformed equation (122) can be found

from the following considerations. By Eq. (149) of Sec. 12 the impedance of our circuit for frequency ω is

$$Z(j\omega) = 2j\omega + 10. \tag{125}$$

In this equation we replace $j\omega$ by p throughout and obtain

$$Z(p) = 2p + 10, \tag{126}$$

the impedance function of p. Since we identify this p with the parameter in Eq. (1), we are chiefly concerned with the values of $Z(p)$ for positive real values of p.

Let $E(p)$ denote the transform of the source voltage $e(t)$, so that in the above problem where $e(t) = 15$, $E(p) = \text{Lap } 15 = 15/p$. Then in view of Eq. (126), Eq. (122) is equivalent to

$$Z(p)I(p) = E(p) \quad\text{or}\quad I(p) = \frac{E(p)}{Z(p)}. \tag{127}$$

This is the Laplace transform analogue of Ohm's law for direct currents, or of Eq. (166) of Sec. 12 for the complex representation of steady-state alternating currents.

For any simple circuit initially dead, $t_0 = 0$ in Eq. (132) of Sec. 11, and the differential equation is

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = e(t). \tag{128}$$

Hence by items 7 and 10 of the Table of Laplace Transforms with $q_0 = 0$ and $i, I, di/dt$ in place of f, F, f' we may write as the transformed equation

$$LpI + R I + \frac{1}{C}\frac{I}{p} = E(p) \quad\text{or}\quad \left(Lp + R + \frac{1}{Cp}\right) I = E(p). \tag{129}$$

But by Eq. (149) of Sec. 12 the impedance for frequency ω is

$$Z(j\omega) = Lj + R + \frac{1}{cj\omega}. \tag{130}$$

Hence the impedance function of p is

$$Z(p) = Lp + R + \frac{1}{Cp}. \tag{131}$$

A comparison of Eqs. (131) and (129) shows that the relation, Eq. (127), holds for any simple circuit.

For simple circuits and the more complicated networks treated in the next section, students to whom steady-state alternating-current theory is second nature will easily obtain the transfer impedance $Z(j\omega)$ appropriate to a given problem, and may then derive $Z(p)$ by replacing $j\omega$ by p as a final step. However, we may put p in place of $j\omega$ at any earlier stage. For example, we may pass directly from the circuit constants to Eq. (131) without using Eq. (130).

As a second example, let it be required to find the current $i(t)$
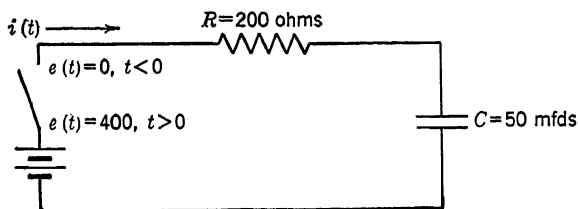
FIG. 53.

in the circuit of Fig. 53, assumed dead at $t = 0$. Here $R = 200$, $C = 50$ microfarads $= 50 \times 10^{-6}$ farad. Hence from Eq. (131)

$$Z(p) = 200 + \frac{10^6}{50p} = \frac{200(p + 100)}{p}. \tag{132}$$

And since $e(t) = 400$, $E(p) = \text{Lap } 400 = 400/p$. Consequently, by Eq. (127),

$$I = \frac{E(p)}{Z(p)} = \frac{400}{p} \frac{p}{200(p + 100)} = \frac{2}{p + 100}. \tag{133}$$

Now by items 1 and 15 the inverse transform is found to be

$$i = 2\epsilon^{-100t}. \tag{134}$$

This is the resulting current.   We note that $i = 2$ when $t = 0$. This equals e/R or $400/200$, because the condenser, having zero charge at time $t = 0$, does not impede the current.   Since $L = 0$ here, Eq. (128) contains no first derivative.   Hence the condition that the circuit be dead imposes no condition on i at $t = 0$, but merely makes $q = 0$ when $t = 0$, already effected by putting the lower limit $t_0 = 0$ in Eq. (128).   For $t = +\infty$, $i = 0$ since the condenser can pass no direct current.

As a third example, let us find the current $i(t)$ in the circuit

$$i(t) \longrightarrow \qquad R = 1000 \text{ ohms}$$
$$e(t) = 0, \ t < 0$$
$$e(t) = 250, \ t > 0 \qquad\qquad C = 2 \text{ mfds}$$
$$L = 1 \text{ henry}$$
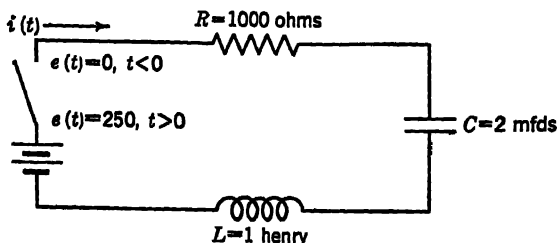
FIG. 54.

of Fig. 54, assumed dead at $t = 0$.   Here $L = 1$, $R = 1,000$, $C = 2 \times 10^{-6}$.   Hence from Eq. (131),

$$Z(p) = p + 1,000 + \frac{10^6}{2p} = \frac{p^2 + 1,000p + 500,000}{p}. \quad (135)$$

Since $e(t) = 250$, $E(p) = \text{Lap } 250 = 250/p$.   Consequently, by Eq. (127),

$$I = \frac{E(p)}{Z(p)} = \frac{250}{p} \frac{p}{p^2 + 1,000p + 500,000}$$
$$= \frac{1}{2} \frac{500}{(p + 500)^2 + 500^2}. \quad (136)$$

Finally by items 1 and 25 the inverse transform is found to be

$$i = \frac{1}{2} \epsilon^{-500t} \sin 500t \quad (137)$$

Thus the transient response of the circuit of Fig. 54 to a sud-

denly applied constant emf is an exponentially damped oscillating current.

56. Networks

It is possible to find i_2, the steady-state alternating-current response in, say, the second element of a network due to e_1 an applied alternating emf of frequency ω in, say, the first element of a network by means of a suitable transfer impedance $A_{12}(j\omega)$ having the property that

$$Z_{12}(j\omega) = \frac{\text{complex } e_1}{\text{complex } i_2} \text{ or } Z_{12}(j\omega) \, (\text{complex } i_2) = (\text{complex } e_1). \quad (138)$$

Such impedances may be found by the method outlined in Sec. 13 or in many cases more efficiently by other means developed in alternating-current network theory. If found as in Sec. 13, $Z_{12}(j\omega)$ results from algebraic operations on equations obtained from the differential equations by replacing d/dt by $j\omega$ and $\int dt$ by $1/(j\omega)$.

If we wished to find the complete transient response $i_2(t)$ to an emf $e_1(t)$ suddenly applied to a dead system, we could start with the same differential equations and take their transforms. To do this, we would replace $e_1(t)$ by $E(p)$, $i_1(t)$, $i_2(t)$, $\cdots$, $i_n(t)$ by $I_1(p)$, $I_2(p)$, $\cdots$, $I_n(p)$, d/dt by p and $\int dt$ by $1/p$. We would then eliminate all the I's except $I_2(p)$. Since this calculation differs from that of the preceding paragraph only in having p in place of $j\omega$, and e_1 and the i's changed to capital letters, the result would be

$$Z_{12}(p)I_2(p) = E_1(p). \quad (139)$$

Thus we may use any convenient means to find $Z_{12}(j\omega)$ for the steady-state alternating-current situation, form $Z_{12}(p)$ from it, and obtain $I_2(p)$ from Eq. (139).

We illustrate the general procedure for the two-mesh network of Fig. 55. If we call the element containing the terminals of the applied emf the first, and seek the resulting transient current in this same element, we shall be concerned with $E_1/I_1 = Z_{11}$ of

simply Z, the *total impedance* or *driving point impedance* of this two-terminal network. To form Z, we consider Z_2 and Z_4 in series, and thus equivalent to $Z_2 + Z_4$. Then by the rule for
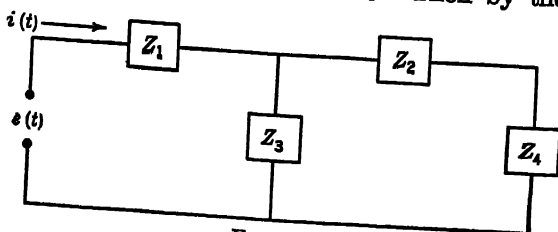
FIG. 55.

parallel impedances derived in Prob. 17 of Exercise V the impedances $(Z_2 + Z_4)$ and Z_3 in parallel combine to give

$$Z^* = \frac{(Z_2 + Z_4)Z_3}{(Z_2 + Z_4) + Z_3} \tag{140}$$

since this makes

$$\frac{1}{Z^*} = \frac{1}{Z_2 + Z_4} + \frac{1}{Z_3}.$$

Finally Z_1 and Z^* in series are equivalent to $Z_1 + Z^*$. Hence the total impedance is

$$Z = Z_1 + \frac{(Z_2 + Z_4)Z_3}{(Z_2 + Z_4) + Z_3} = \frac{(Z_2 + Z_4)(Z_1 + Z_3) + Z_1 Z_3}{Z_2 + Z_4 + Z_3}. \tag{141}$$

For use in a steady-state alternating-current calculation, as in Eq. (138), we would need $Z(j\omega)$. This could be found by writing the analogues of Eq. (130)

$$Z_1(j\omega) = L_1 j\omega + R_1 + \frac{1}{C_1 j\omega}, \tag{142}$$

and the similar expressions for Z_2, Z_3, Z_4 in terms of the appropriate lumped constants. Substitution of these in Eq. (141) gives $Z(j\omega)$.

To find the Laplace transform of a transient response, as in

Eq. (139), we need $Z(p)$. We make use of the analogues of Eq. (131)

$$Z_1(p) = L_1 p + R_1 + \frac{1}{C_1 p}, \qquad (143)$$

and the similar equations for Z_2, Z_3, Z_4 in terms of the appropriate lumped constants. Substitutions of these in Eq. (141) gives $Z(p)$. Since our problem involves the emf $e_1(t)$ and the current $i_1(t)$ in the first element only, we drop the subscripts 1, and in place of $Z_{11}(p)I_1(p) = E_1(p)$ write

$$Z(p)I(p) = E(p) \qquad \text{or} \qquad I(p) = \frac{E(p)}{Z(p)}. \qquad (144)$$

As a concrete example, let us find the transient current in the
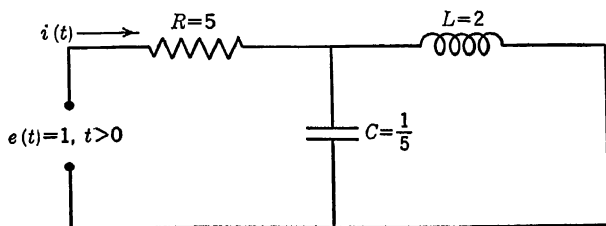
Fig. 56. Prototype network.

prototype network of Fig. 56, differing from an actual network by a scale factor, say 10^6. From Eq. (143) we have here

$$Z_1 = 5, \qquad Z_2 = 2p, \qquad Z_3 = \frac{5}{p}. \qquad (145)$$

Z_2 and Z_3 combine in parallel to give

$$Z^* = \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{2p \dfrac{5}{p}}{2p + \dfrac{5}{p}} = \frac{10p}{2p^2 + 5}. \qquad (146)$$

And this in series with $Z_1 = 5$ gives

$$Z = 5 + \frac{10p}{2p^2 + 5} = \frac{10p^2 + 10p + 25}{2p^2 + 5} \qquad (147)$$

Since $e(t) = 1$, by item 14, $E(p) = 1/p$. Hence from Eq. (144)

$$I = \frac{E}{Z} = \frac{1}{p} \frac{2p^2 + 5}{10p^2 + 10p + 25} = \frac{1}{10} \frac{2p^2 + 5}{p(p^2 + p + 2.5)}. \quad (148)$$

By the methods explained in Sec. 54, we may deduce that

$$\frac{2p^2 + 5}{p(p^2 + p + 2.5)} = \frac{2}{p} - \frac{2}{p^2 + p + 2.5} = 2\frac{1}{p}$$

$$- \frac{4}{3} \frac{1.5}{(p + 0.5)^2 + 1.5^2}$$

or

$$= 2\frac{1}{p} + 2 \operatorname{Im} \left(-\frac{2}{3} \right) \frac{1}{p + 0.5 + 1.5i}. \quad (149)$$

From items 1 and 14 for the first term and 25 or 15 for the second term, we may find the inverse transform of the last two expressions. Hence from either form, and Eq. (148), we find for $i(t) = \operatorname{Lap}^{-1} I$,

$$i = \tfrac{1}{5} - \tfrac{2}{15}\epsilon^{-0.5t} \sin 1.5t. \quad (150)$$

For the actual network with the resistance still 5 ohms, but inductance 2×10^{-6} henry and capacitance $\tfrac{1}{5} \times 10^{-6}$ farad, the response to an emf of 1 volt suddenly applied would be

$$i = \tfrac{1}{5} - \tfrac{2}{15}\epsilon^{-5\times10^{6}t} \sin 1.5 \times 10^6 t. \quad (151)$$

This is obtained from Eq. (150) by multiplying the coefficients of t by 10^6.

EXERCISE XXX

Assume that the circuit of Fig. 57 is dead at time $t = 0$, and find the transient current response to a suddenly applied emf.

1. $e(t) = 3$, if $R = 2$, $L = 1$, no capacitance present.
2. $e(t) = e_0$, if $R = R$, $L = L$, no capacitance present.
3. $e(t) = 10$, if $R = 4$, $C = 10^{-5}$, no inductance present.
4. $e(t) = e_0$, if $R = R$, $C = C$, no inductance present.
5. $e(t) = 500$, if $R = 400$, $L = 1$, $C = \tfrac{1}{3} \times 10^{-4}$.
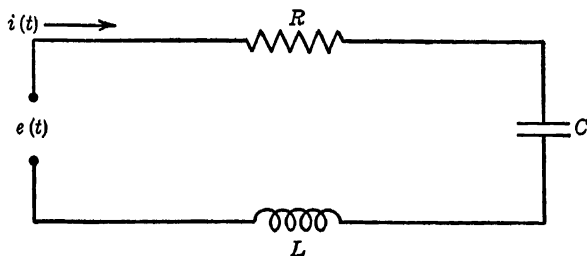
Fig. 57.

6. $e(t) = 100$, if $R = 100$, $L = 1$, $C = 8 \times 10^{-5}$.

7. $e(t) = \sin 20t$, if $L = 2$, $C = 2 \times 10^{-4}$, no resistance present.

8. $e(t) = \sin 10t$, if $R = 400$, $L = 10$,
   $C = \frac{1}{3} \times 10^{-3}$.

9. $e(t) = \sin 2t$, if $R = 100$, $L = 5$,
   $C = 4 \times 10^{-4}$.

Use the initial value theorem of Prob. 54 of Exercise XXVIII

10. And Eq. (123) to show that $i(0) = 0$ in Eq. (124).

11. And Eq. (133) to show that $i(0) = 2$ in Eq. (134).

12. The motion of the mass in Fig. 58 is governed by Eq. (135) of Sec. 11. Let

$$F = f(t), \quad F(p) = \text{Lap } f(t)$$

and

$$S(p) = \text{Lap } s(t).$$

Fig. 58.   Vibrating mass.

Let the mass be initially at rest, and let $s(0) = 0$. Show that if $A(p) = mp^2 + \beta p + k$, then $Z(p)S(p) = F(p)$.

Use Prob. 12 to find $Z(p)$ when the mass weighs $2g$ lb. so that $m = 2$, $\beta = 8$, and $k = 26$. For this case determine $s(t)$ when the suddenly applied force $F = f(t)$ is

13. $f(t) = 4$.   **14.** $f(t) = 4 \sin 2t$.
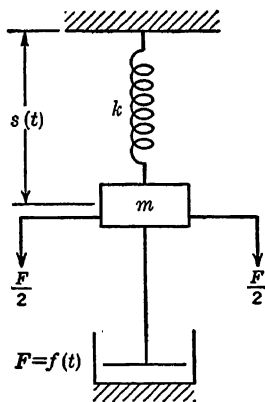
15. The oscillatory motion of the disk in Fig. 59 is governed by Eq. (138) of Sec. 11.   Let $M = f(t)$, $F(p) = \text{Lap } f(t)$ and
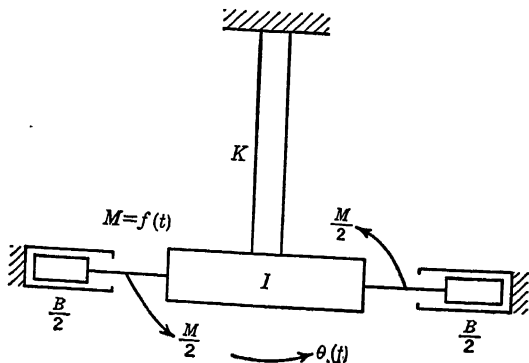
FIG. 59.   Oscillating disk.

$\Theta(p) = \text{Lap } \theta(t)$.   Let the disk be initially at rest, and let $\theta(0) = 0$.   Show that if

$$Z(p) = Ip^2 + Bp + K, \text{ then } Z(p)\Theta(p) = F(p).$$

Use Prob. 15 to find $Z(p)$ when $I = 1$ slug-ft.^{2}, $B = 4$, and $K = 5$.   For this case determine $\theta(t)$ when there is suddenly applied

16. A torque $M = f(t) = 6$.

17. A unit impulsive torque.   That is, $\int_0^t f(t)dt = 1$.   Hence $\frac{1}{p} F(p) = \text{Lap } 1 = \frac{1}{p}$, and $F(p) = 1$.

In each case assume the indicated circuit initially dead, and find the transient current response in the first element to the given applied emf.

18. $e(t) = 10$ in the circuit of Fig. 60.
19. $e(t) = 10 \sin 2t$ in the circuit of Fig. 60.
20. $e(t) = 100$ in the circuit of Fig. 61.
21. $e(t) = 100 \sin 10t$ in the circuit of Fig. 61.
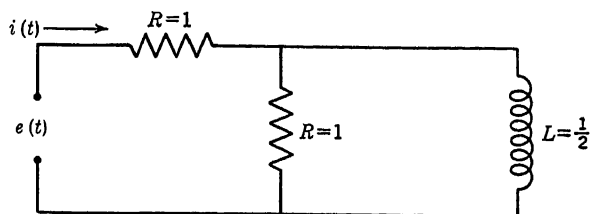
FIG. 60.
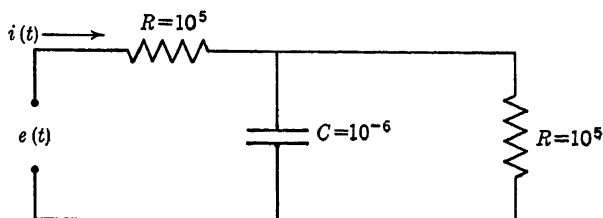
FIG. 61.
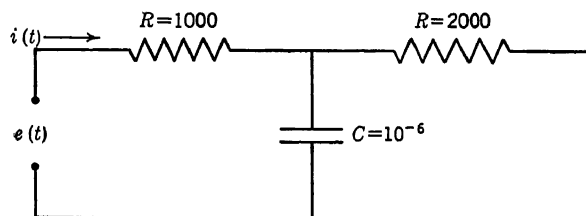
FIG. 62.
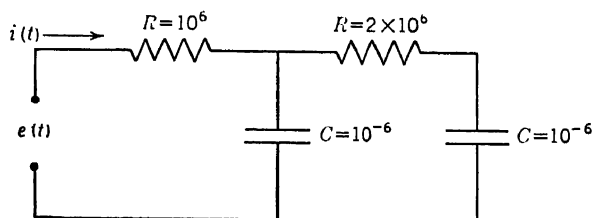
FIG. 63.

22. $e(t) = 750$ in the circuit of Fig. 62.
23. $e(t) = 100$ in the circuit of Fig. 63.

57. Partial Derivatives

The Laplace transform may be used to solve certain problems involving partial differential equations. Suppose that there are two independent variables x, a distance, and t, the time. Consider a dependent variable $f(x,t)$ which is a function of x and t. Then $F(x,p)$, the time *Laplace transform* of $f(x,t)$ is defined by the equation

$$F(x,p) = \int_0^\infty e^{-pt} f(x,t) dt. \tag{152}$$

In the integration x is treated as a constant. We express the relation of the new function of x and p to the old function of x and t by writing

$$F(x,p) = \text{Lap } f(x,t) \qquad \text{or} \qquad f(x,t) = \text{Lap}^{-1} F(x,p), \tag{153}$$

In forming the partial derivative with respect to t, $\dfrac{\partial f}{\partial t}$, we keep x fixed. But for x fixed, Eq. (152) is identical with Eq. (1). Hence by arguing as in Eq. (19), we may deduce that

$$\text{Lap} \frac{\partial f}{\partial t} = -f(x,0+) + p \, \text{Lap} f(x,t). \tag{154}$$

And for a system dead initially, $f(x,0+) = 0$ and

$$\text{Lap} \frac{\partial f}{\partial t} = p \, \text{Lap} f(x,t) = pF(x,p). \tag{155}$$

Similarly the transform of any higher partial derivative with respect to t, or of the integral with respect to t, may be found by slightly modifying the notation in items 2 to 9 of the Table of Laplace Transforms. In particular, for initially dead systems, *partial differentiation* of the function $f(x,t)$ *with respect to t* corresponds to *multiplication* of the transform $F(x,p)$ *by p*.

The situation is different when we form $\dfrac{\partial f}{\partial x}$. In fact, *partial*

differentiation of the function $f(x,t)$ *with respect to* x, keeping t fixed, corresponds to *partial differentiation* of the transform $F(x,p)$ *with respect to* x, keeping p fixed. To see this, we calculate $\dfrac{\partial F}{\partial x}$ from Eq. (152). Assuming that $f(x,t)$ and $\dfrac{\partial f}{\partial x}$ are regular and do not increase too rapidly for large values of t, we may carry out the differentiation on the right inside the integral sign and thus obtain

$$\frac{\partial F}{\partial x} = \int_0^\infty e^{-pt} \frac{\partial f}{\partial x} \, dt. \tag{156}$$

Since this differs from Eq. (152) only in having $\dfrac{\partial F}{\partial x} = F_x(x,p)$ in place of $F(x,p)$ and $\dfrac{\partial f}{\partial x} = f_x(x,t)$ in place of $f(x,t)$, it shows that

$$\text{Lap } f_x(x,t) = F_x(x,p) \qquad \text{or} \qquad \text{Lap } \frac{\partial f}{\partial x} = \frac{\partial F}{\partial x}. \tag{157}$$

This shows that differentiation with respect to x on the function corresponds to a similar operation on the transform.

As illustrations of the principles just derived, assuming a system initially dead, we have for the second derivatives

$$\text{Lap } \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 F}{\partial x^2}, \qquad \text{Lap } \frac{\partial^2 f}{\partial x \, \partial t} = p \frac{\partial F}{\partial x}, \qquad \text{Lap } \frac{\partial^2 f}{\partial t^2} = p^2 F. \tag{158}$$

We turn now to specific examples. Our first problem is to find the solution of the partial differential equation

$$\frac{\partial f}{\partial t} = 8xt + 2 \sin x \tag{159}$$

which satisfies the initial condition

$$f(x,0) = x^2. \tag{160}$$

By substituting from Eq. (160) into the analogue of item 2 or Eq. (154) we may determine the transform of $\dfrac{\partial f}{\partial t}$. And we may find

the transform of the right member of Eq. (159) by treating x as a constant and using items 1, 16, and 14. Thus the transform of Eq. (159) is

$$-x^2 + pF = \frac{8x}{p^2} + \frac{2 \sin x}{p} \quad \text{or}$$

$$F = 8x \frac{1}{p^3} + 2 \sin x \frac{1}{p^2} + x^2 \frac{1}{p}. \qquad (161)$$

We next find $f(x,t) = \text{Lap}^{-1} F(x,p) = \text{Lap}^{-1} F$ by using items 1, 17, 16, and 14. This gives the solution of our problem

$$f(x,t) = 4xt^2 + 2t \sin x + x^2.$$

As a second problem let us find the solution of the equation

$$\frac{\partial^2 f}{\partial x \, \partial t} = 8xt \qquad (162)$$

which satisfies the initial and boundary conditions

$$f(x,0) = x^2, \qquad f(0,t) = 3t. \qquad (163)$$

To find the transform of $\dfrac{\partial^2 f}{\partial x \, \partial t}$ we first substitute $f(x,0+) = x^2$ in Eq. (154) and then replace the f in Eq. (157) by $\dfrac{\partial f}{\partial t}$. And as in Eq. (161), Lap $8xt = 8x/p^2$. Thus the transform of Eq. (162) is

$$-2x + p \frac{\partial F}{\partial x} = \frac{8x}{p^2}, \quad \text{or} \quad \frac{\partial F}{\partial x} = \frac{8x}{p^3} + \frac{2x}{p}. \qquad (164)$$

Since this involves no differentiation with respect to p, it is essentially an ordinary differential equation in F and x. Hence we may proceed as in Sec. 27. Integrating with respect to x keeping p constant, we obtain

$$F(x,p) = \frac{4x^2}{p^3} + \frac{x^2}{p} + G(p). \qquad (165)$$

Since $F(x,p) = \text{Lap } f(x,t)$, $F(0,p) = \text{Lap } f(0,t) = \text{Lap } 3t$ by Eq.

(163). Hence from items 1 and 16, $F(0,p) = 3/p^2$. But from Eq. (165) we find $F(0,p) = G(p)$, so that

$$G(p) = \frac{3}{p^2} \quad \text{and} \quad F = 4x^2\frac{1}{p^3} + x^2\frac{1}{p} + 3\frac{1}{p^2}. \quad (166)$$

We next use items 1, 17, 14, and 16 to find $\text{Lap}^{-1} F$. The result is

$$F(x,t) = 2x^2t^2 + x^2 + 3t. \quad (167)$$

This is the solution of Prob. 2, or Eqs. (162) and (163).

The partial differential equation of the first order

$$2\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \quad (168)$$

has a unique solution assuming any prescribed set of values on the broken line consisting of the positive half of the t axis and the positive half of the x axis. Suppose we demand that

$$u(x,0) = 0 \quad \text{for} \quad x > 0 \quad \text{and} \quad u(0,t) = 3t^2 \quad \text{for } t > 0. \quad (169)$$

As our third problem, let us find the solution of Eq. (168) which satisfies the boundary conditions of Eq. (169). Since $u(x,0+) = 0$, we may use the analogue of item 7 or Eq. (155) to determine the transform of $\frac{\partial u}{\partial t}$. And Eq. (157) gives the transform of $\frac{\partial u}{\partial x}$. In each case we replace f, F by u, U. Thus the transform of Eq. (168) is

$$2pU + \frac{\partial U}{\partial x} = 0 \quad \text{or} \quad \frac{\partial U}{\partial x} = -2pU. \quad (170)$$

To solve this we proceed as in Prob. 21 of Exercise XIII. Treating it as essentially an ordinary linear differential equation with constant coefficients we find the solution

$$U(x,p) = A(p)e^{-2px}. \quad (171)$$

Since $U(x,p) = \text{Lap } u(x,t)$, $U(0,p) = \text{Lap } u(0,t) = \text{Lap } 3t^2$ by Eq. (169). Hence from items 1 and 17, $U(0,p) = 6/p^3$. But

from Eq. (171) we find $U(0,p) = A(p)$, so that

$$A(p) = \frac{6}{p^3} \qquad \text{and} \qquad U = 6e^{-(2x)p}\frac{1}{p^3}. \qquad (172)$$

We next use items 1, 12, and 17 to find Lap^{-1} U.   The result is

$$u(x,t) = 0 \qquad \text{for } t < 2x$$
$$u(x,t) = 3(t - 2x)^2 \qquad \text{for } t > 2x. \qquad (173)$$

This is the solution of our problem.

Let us define $g(z) = 0$ for $z < 0$ and $g(z) = 3z^2$ for $z > 0$. Then $u(x,t) = g(t - 2x)$, since if $z = t - 2x$, $t < 2x$ when $z < 0$ and $t > 2x$ when $z > 0$.   But $(t - 2x) = -2(x - t/2)$.   Hence from the discussion of $f(x - vt)$ in Sec. 31 in connection with Eq. (81) it follows that the $u(x,t)$ of Eq. (173) represents a wave traveling to the right with velocity $\frac{1}{2}$.

As our fourth problem, let us find the solution of the equation

$$2\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -4u \qquad (174)$$

which on the half axes satisfies the boundary conditions

$$u(x,0) = 0 \quad \text{for } x > 0 \quad \text{and} \quad u(0,t) = 3t^2 \quad \text{for } t > 0. \qquad (175)$$

If $U = \text{Lap } u$, by item 1, Lap $(-4u) = -4U$.   And the transform of the left member of Eq. (174) is found as in Eq. (170). Thus we find

$$2pU + \frac{\partial U}{\partial x} = -4U, \qquad \text{or} \qquad \frac{\partial U}{\partial x} = -(4 + 2p)U. \qquad (176)$$

Solving this as we did Eq. (170), we find

$$U(x,p) = A(p)e^{-(4+2p)x}. \qquad (177)$$

From Eqs. (177) and (175) we find that

$$A(p) = U(0,p) = \text{Lap } u(0,t) = \text{Lap } 3t^2 = 6/p^3.$$

By substituting this in Eq. (177), we obtain

$$U = \frac{6}{p^3} e^{-4x-2px} = 6e^{-4x}e^{-(2x)p}\frac{1}{p^3}. \qquad (178)$$

To find Lap^{-1} U, we treat x as constant and use items 1, 12 and 17. This gives as the solution of Prob. 4, or Eqs. (174) and (175),

$$u(x,t) = 0 \qquad \text{for } t < 2x$$
$$u(x,t) = 3e^{-4x}(t - 2x)^2 \qquad \text{for } t > 2x. \tag{179}$$

Let us define $g(z) = 0$ for $z < 0$ and $g(z) = 3e^{2x}z^2$ for $z > 0$. Then $u(x,t) = e^{-2t}g(t - 2x)$. Since $t - 2x = -2\left(x - \dfrac{t}{2}\right)$, by the property of $f(x - vt)$ referred to above $g(t - 2x)$ represents a wave traveling to the right with velocity $\frac{1}{2}$. And the solution $u(x,t)$ is this wave modulated with the damping factor e^{-2t}.

58. The Lossless Transmission Line

The equations governing the flow of electricity in a long lossless transmission line are

$$-\frac{\partial e}{\partial x} = L\frac{\partial i}{\partial t}, \qquad -\frac{\partial i}{\partial x} = C\frac{\partial e}{\partial t}. \tag{180}$$

There is a third relation derivable from these

$$\frac{\partial^2 e}{\partial x^2} = LC\frac{\partial^2 e}{\partial t^2} \quad \text{or} \quad \frac{\partial^2 e}{\partial t^2} = v^2\frac{\partial^2 e}{\partial x^2}, \quad \text{if } v = \frac{1}{\sqrt{LC}}. \tag{181}$$

Compare Eq. (97) of Sec. 33 and Eqs. (102) and (103) of Sec. 44. We recall that x represents distance along the line in miles and t represents the time in seconds. The potential $e(x,t)$ volts and the current $i(x,t)$ amperes are each function of position and time or of x and t like the function $f(x,t)$ of Eq. (152) and thus have time Laplace transforms $E(x,t)$ and $I(x,t)$ defined by relations similar to Eq. (152). Here the series inductance of L henrys per mile and the shunt capacitance of C farads per mile are thought of as uniformly distributed. They are thus unlike the corresponding letters in Eqs. (131) and (143) which represented lumped parameters and had the units of henrys and farads.

We shall confine our applications to systems initially dead, so that in transforming time derivatives we may use the analogues of items 7 and 8, or Eqs. (155) and (158). For space derivatives

we use Eqs. (157) and (158). Thus on taking transforms of Eq. (180) we find

$$- \frac{\partial E}{\partial x} = LpI, \qquad - \frac{\partial I}{\partial x} = CpE. \qquad (182)$$

And the transform of Eq. (181) is

$$\frac{\partial^2 E}{\partial x^2} = LCp^2 E \qquad \text{or} \qquad \frac{\partial^2 E}{\partial x^2} = \frac{p^2}{v^2} E. \qquad (183)$$

Note that the first form might have been obtained from Eq. (182) by differentiating the first relation partially with respect to x and substituting for $\frac{\partial I}{\partial x}$ from the second relation.

To solve Eq. (183) we proceed as in Prob. 25 of Exercise XIII. Treating it as essentially an ordinarily differential equation with constant coefficients, we find the solution

$$E(x,p) = A(p)\epsilon^{-px/v} + B(p)\epsilon^{px/v}. \qquad (184)$$

By differentiating this relation with respect to x, we find

$$\frac{\partial E}{\partial x} = - \frac{p}{v} A(p)\epsilon^{-px/v} + \frac{p}{v} B(p)\epsilon^{px/v}. \qquad (185)$$

We may combine this with the first relation of Eq. (182) to deduce

$$I = - \frac{1}{Lp} \frac{\partial E}{\partial x} = \frac{1}{Lv} [A(p)\epsilon^{-px/v} - B(p)\epsilon^{px/v}]. \qquad (186)$$

In Prob. 7 of Exercise XVIII we defined the *characteristic impedance* of a transmission line with constants R, L, G, C as $Z_K = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$. For the lossless line considered here, $R = 0$, $G = 0$ so that the characteristic impedance reduces to

$$Z_K = \sqrt{\frac{L}{C}}.$$

But by Eq. (181), $v = 1/\sqrt{LC}$. Hence $Lv = \sqrt{L/C} = Z_K$ and

Eq. (186) may be written

$$I(x,p) = \frac{A(p)\epsilon^{-px/v} - B(p)\epsilon^{px/v}}{Z_K} \tag{187}$$

If $e(x,t)$ and $i(x,t)$ satisfy Eq. (180) and the system is initially dead, their Laplace transforms are given by Eqs. (184) and (187).

59. The Infinite Line

Suppose that we apply the results of Sec. 58 to a line which is of infinite length. We seek solutions $e(x,t)$ and $i(x,t)$ which for any fixed t remain finite as $x \rightarrow +\infty$. Hence the transforms $E(x,p)$ and $I(x,p)$ for any fixed p will remain finite as $x \rightarrow +\infty$. But for any fixed p, we have

$$\lim_{x \rightarrow +\infty} \epsilon^{-px/v} = 0, \quad \lim_{x \rightarrow +\infty} \epsilon^{px/v} = +\infty$$

This shows that we must take $B(p) = 0$ in Eq. (184) to make $E(x,p)$ finite as $x \rightarrow +\infty$. With $B(p) = 0$, Eqs. (184) and (187) become

$$E(x,p) = A(p)\epsilon^{-px/v}, \tag{188}$$

and

$$I(x,p) = \frac{A(p)}{Z_K} \epsilon^{-px/v} = \frac{E(x,p)}{Z_K}. \tag{189}$$

From item 1 the inverse of the last relation is

$$i(x,t) = \frac{e(x,t)}{Z_K}. \tag{190}$$

Thus for an infinite line the knowledge of $e(x,t)$ is practically a complete solution since to determine $i(x,t)$ from it we merely have to divide by the characteristic impedance $Z_K = \sqrt{L/C}$.

In particular, let us consider the infinite line of Fig. 64. Since $L = 0.8$ and $C = 2 \times 10^{-7}$, we find for the propagation velocity v and the characteristic impedance Z_K the following values:

$$v = \frac{1}{\sqrt{LC}} = 2{,}500 \quad \text{and} \quad Z_K = \sqrt{\frac{L}{C}} = 2{,}000. \tag{191}$$

The lumped parameters of the first element, $L_1 = 2,000$ henrys and $R_1 = 4,000$ ohms in series, are equivalent to an impedance

$$Z_1(p) = 2,000p + 4,000 \qquad (192)$$

by Eq. (143).   We wish to find the transient potential of the line $e(x,t)$ in response to the suddenly impressed emf $e_i(t) = 240$.   To do this, we consider the voltage drop across the first element MN
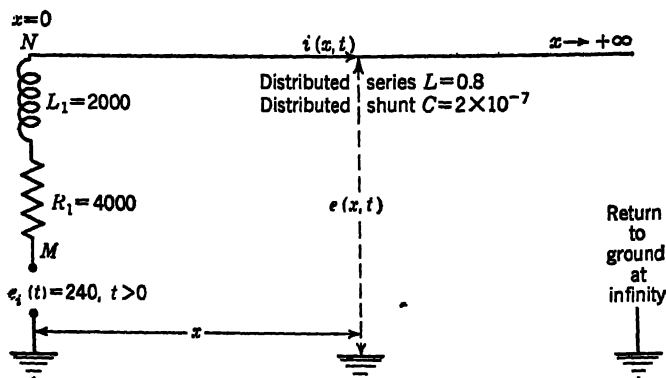
Fig. 64.   Infinite lossless transmission line.

in two ways.   We first observe that this drop $e_1(t)$ is the difference between the potential at M, $e_i(t)$ and the potential at N, $e(0,t)$. Hence

$$e_1(t) = e_i(t) - e(0,t) = 240 - e(0,t). \qquad (193)$$

By items 1 and 14, this leads to the transformed equation

$$E_1(p) = \frac{240}{p} - E(0,p). \qquad (194)$$

The current passing through the first element is $i(0,t)$.   Its transform $I(0,p)$ is related to $E_1(p)$ and $Z_1(p)$ by the Laplace transform analogue of Ohm's law, Eq. (127).   From this and Eq. (192) we find

$$E_1(p) = Z_1(p)I(0,p) = (2,000p + 4,000)I(0,p). \qquad (195)$$

We next put $Z_K = 2,000$ from Eq. (191) into Eq. (189) to obtain

$$I(x,p) = \frac{E(x,p)}{2,000} \quad \text{and} \quad I(0,p) = \frac{E(0,p)}{2,000}. \quad (196)$$

Substitution in Eq. (195) and comparison with Eq. (194) shows that

$$(2,000p + 4,000)\frac{E(0,p)}{2,000} = \frac{240}{p} - E(0,p). \quad (197)$$

It follows that

$$(p + 3)E(0,p) = \frac{240}{p}, \qquad E(0,p) = \frac{240}{p(p + 3)}$$

$$= 80\left(\frac{1}{p} - \frac{1}{p + 3}\right). \quad (198)$$

But with $x = 0$, Eq. (188) gives $E(0,p) = A(p)$. Consequently

$$E(x,p) = E(0,p)\epsilon^{-px/v} = 80\epsilon^{-(x/v)p}\left(\frac{1}{p} - \frac{1}{p + 3}\right). \quad (199)$$

To find $\text{Lap}^{-1} E(x,p)$, we treat x as a constant and use items 1, 12, 14, and 15. This gives as the solution for $e(x,t)$

$$e(x,t) = 0 \quad \text{for } t < \frac{x}{v}, \quad \text{where } v = 2,500,$$

$$e(x,t) = 80[1 - \epsilon^{-3(t-x/v)}] \quad \text{for } t > \frac{x}{v}. \quad (200)$$

Let us define $g(z) = 0$ for $z < 0$ and $g(z) = 80(1 - \epsilon^{-3z})$ for $z > 0$. Then $e(x,t) = g(t - x/v)$, since if $z = t - (x/v)$, $t < x/v$ when $z < 0$ and $t > x/v$ when $z > 0$. But

$$t - \left(\frac{x}{v}\right) = -\frac{1}{v}(x - vt).$$

Hence from the discussion of $f(x - vt)$ in connection with Eq. (81) of Sec. 31 it follows that the $e(x,t)$ of Eq. (200) represents a wave traveling to the right with velocity $v = 2,500$. In particular, consider a time t_1. For any x_2 between 0 and vt_1, $0 < x_2 < vt_1$,

so that $t_1 > x_2/v$. Hence $e(x_2,t_1)$ is determined from the second relation of Eq. (200) and is positive. But for any x_3 beyond vt_1, $x_3 > vt_1$, so that $t_1 < x_3/v$. Hence $e(x_3,t_1)$ is determined from the first relation of Eq. (200) and is zero. Thus at t_1 sec. after closing the switch at $t = 0$, the effect has made itself felt for vt_1 miles down the line. Since vt_1 is the distance that would be traveled by a point starting at the origin and moving to the right with velocity v, we may say that the effect of closing the switch
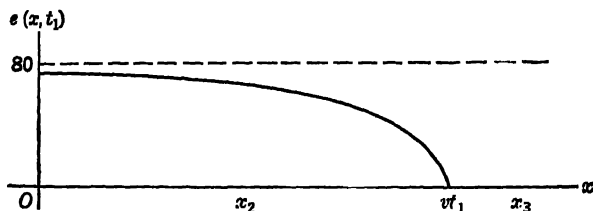
Fig. 65.

covers distance at velocity $v = 2,500$ miles per second [see Fig. 65 for a graph of $e(x,t_1)$].

More generally, let us consider an infinite line with constants v and Z_K. Suppose that the home end, $x = 0$, is connected to the ground through an element, or combination of elements, having a total series impedance $Z_1(p)$, in series with an impressed voltage $e_i(t)$. Then corresponding to Eqs. (193) and (194) we now have

$$e_1(t) = e_i(t) - e(0,t) \qquad (201)$$

and

$$E_1(p) = E_i(p) - E(0,p). \qquad (202)$$

From Eq. (189) we obtain

$$I(x,p) = \frac{E(x,p)}{Z_K} \qquad \text{and} \qquad I(0,p) = \frac{E(0,p)}{Z_K}. \qquad (203)$$

We substitute this into the equation similar to Eq. (195)

$$E_1(p) = Z_1(p)I(0,p) = Z_1(p)\frac{E(0,p)}{Z_K}. \qquad (204)$$

A comparison with Eq. (202) shows that

$$Z_1(p) \frac{E(0,p)}{Z_K} = E_i(p) - E(0,p). \tag{205}$$

By solving this for $E(0,p)$, we find

$$E(0,p) = \frac{Z_K E_i(p)}{Z_1(p) + Z_K}. \tag{206}$$

For given values of Z_K, $Z_1(p)$, and $e_i(t)$ determining

$$E_i(p) = \text{Lap } e_i(t),$$

$E(0,p)$ becomes a known function of p whose inverse transform may be found by the method of Sec. 54. This gives

$$e(0,t) = \text{Lap}^{-1} E(0,p).$$

Let us define $g(z) = 0$ for $z < 0$ and $g(z) = e(0,z)$ for $z > 0$. Then, as in Eq. (199), from Eq. (188) we find

$$E(x,p) = E(0,p)\epsilon^{-px/v} = \epsilon^{-(x/v)p} \text{ Lap } g(t). \tag{207}$$

We may now obtain $\text{Lap}^{-1} E(x,p)$ from item 12 as

$$e(x,t) = g\left(t - \frac{x}{v}\right). \tag{208}$$

Since $t - (x/v) = -(1/v)(x - vt)$, this represents a wave traveling to the right with velocity v. At t_1 sec. the effect has made itself felt for vt_1 miles down the line, since if $x_3 > vt_1$, $vt_1 - x_3 < 0$ and $[t_1 - (x_3/v)] < 0$, so that $e(x_3,t_1) = 0$.

We may substitute from Eq. (206) into Eq. (203) to deduce that

$$I(0,p) = \frac{E_i(p)}{Z_1(p) + Z_K}. \tag{209}$$

The similarity of this relation to Eq. (144) shows that $i(0,t)$, the current through the impedance $Z_1(p)$ in series with the infinite line due to the impressed voltage $e_i(t)$ is the same as the current through the impedance $Z_1(p)$ in series with a lumped resistance $R = Z_K$ due to an impressed voltage $e_i(t)$.

60. The Finite Line

We shall now apply the results of Sec. 58 to a finite line of length S miles. We recall that

$$v = \frac{1}{\sqrt{LC}}, \quad \text{and} \quad Z_K = \sqrt{\frac{L}{C}}. \quad (210)$$

And in terms of these constants we found in Eqs. (184) and (187) that

$$E(x,p) = A(p)\epsilon^{-px/v} + B(p)\epsilon^{px/v}, \quad (211)$$

$$I(x,p) = \frac{A(p)\epsilon^{-px/v} - B(p)\epsilon^{px/v}}{Z_K}. \quad (212)$$

Let us consider the finite line of Fig. 66. Here there is no
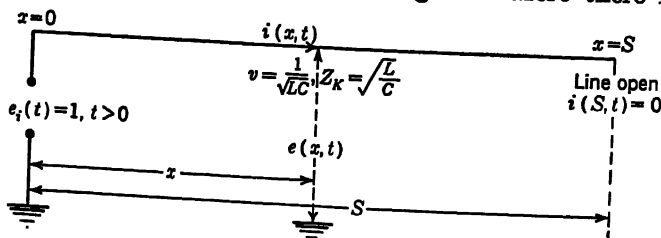
FIG. 66. Lossless transmission line of length S.

impedance in series with the line, and the impressed voltage at the home end $e_i(t) = 1$. Hence

$$e(0,t) = e_i(t) = 1 \quad \text{and} \quad E(0,p) = \frac{1}{p}. \quad (213)$$

As the line is open at the far end, $x = S$, no current flows there. Hence

$$i(S,t) = 0 \quad \text{and} \quad I(S,p) = 0. \quad (214)$$

We may determine $A(p)$ and $B(p)$ from the last two conditions. To do this, first put $x = 0$ in Eq. (211) and use Eq. (213). The result is

$$\frac{1}{p} = A(p) + B(p). \quad (215)$$

Then put $x = S$ in Eq. (212), and use Eq. (214). The result is

$$0 = \frac{1}{Z_K} [A(p)\epsilon^{-pS/v} - B(p)\epsilon^{pS/v}]. \tag{216}$$

We solve Eqs. (215) and (216) as simultaneous in $A(p)$, $B(p)$. From Eq. (216),

$$A(p)\epsilon^{-pS/v} = B(p)\epsilon^{pS/v} \quad \text{and} \quad B(p) = A(p)\epsilon^{-2pS/v}. \tag{217}$$

Substitution of this in Eq. (215) leads to

$$\frac{1}{p} = A(p)[1 + \epsilon^{-2pS/v}] \quad \text{and} \quad A(p) = \frac{1}{p(1 + \epsilon^{-2pS/v})}. \tag{218}$$

We may now substitute this expression in Eq. (217) to obtain

$$B(p) = \frac{\epsilon^{-2pS/v}}{p(1 + \epsilon^{-2pS/v})}. \tag{219}$$

Let us now use the values of $A(p)$ and $B(p)$ just found in Eqs. (218) and (219) in Eqs. (211) and (212). The results may be written

$$E(x,p) = \frac{1}{p} \frac{1}{1 + \epsilon^{-2pS/v}} [\epsilon^{-px/v} + \epsilon^{(x-2S)p/v}], \tag{220}$$

$$I(x,p) = \frac{1}{Z_K} \frac{1}{p} \frac{1}{1 + \epsilon^{-2pS/v}} [\epsilon^{-px/v} - \epsilon^{(x-2S)p/v}]. \tag{221}$$

The inverse transforms of these may be found by using a suitable series expansion. We begin with the Maclaurin's series

$$\frac{1}{1 + y} = 1 - y + y^2 - \cdots + (-1)^n y^n + \cdots. \tag{222}$$

This is valid for $|y| < 1$, and is essentially the result used to find the sum of an infinite geometric series in elementary algebra. Since

$$\frac{2pS}{v} > 0, \quad y = \epsilon^{-2pS/v} \text{ makes } 0 < y < 1 \text{ and } |y| < 1. \tag{223}$$

Hence we may use this as the y of Eq. (222) and derive

$$\frac{1}{1 + \epsilon^{-2pS/v}} = 1 - \epsilon^{-2pS/v} + \epsilon^{-4pS/v} - \epsilon^{-6pS/v} + \cdots. \quad (224)$$

Substitution of this in Eq. (220) leads t

$$E(x,p) = \frac{1}{p}(1 - \epsilon^{-2pS/v} + \cdots)[\epsilon^{-px/v} + \epsilon^{(x-2S)p/v}]$$

$$= \epsilon^{-(x/v)p}\frac{1}{p} + \epsilon^{-[(2S-x)/v]p}\frac{1}{p} - \epsilon^{-[(2S+x)/v]p}\frac{1}{p}$$

$$- \epsilon^{-[(4S-x)/v]p}\frac{1}{p} + \epsilon^{-[(4S+x)/v]p}\frac{1}{p} \cdots. \quad (225)$$

Let us introduce the symbol $1(z)$ to represent the *unit-step function* of z defined by

$$1(z) = 0 \quad \text{for } z < 0$$
$$1(z) = 1 \quad \text{for } z > 0. \quad (226)$$

Then from items 12 and 14 it follows that

$$\text{Lap}^{-1}\, \epsilon^{-bp}\frac{1}{p} = 1(t - b). \quad (227)$$

The function $1(t - b)$ is a unit-step function of $(t - b)$, or a *delayed unit-step function* of t, with delay or lag b. That is, it is zero until $t = b$, and from then on it equals 1.

By item 1, $\text{Lap}^{-1} E(x,p)$ may be found by suitably combining the inverse transforms of the individual terms of the series in Eq. (225). And these may be found by using Eq. (227) with $b = x/v$, $(2S - x)/v$, etc. Thus we find

$$e(x,t) = 1\left(t - \frac{x}{v}\right) + 1\left(t - \frac{2S - x}{v}\right) - 1\left(t - \frac{2S + x}{v}\right)$$

$$- 1\left(t - \frac{4S - x}{v}\right) + 1\left(t - \frac{4S + x}{v}\right) + \cdots. \quad (228)$$

The lags in these terms depend on x, but for interior points of the

line $0 < x < S$ so that

$$0 < \frac{x}{v} < \frac{S}{v}, \qquad \frac{S}{v} < \frac{2S - x}{v} < \frac{2S}{v},$$

$$\frac{2S}{v} < \frac{2S + x}{v} < \frac{3S}{v}, \cdot \cdot \cdot \cdot \quad (229)$$

It follows that for any time t_1 such that $0 < t_1 < S/v$, only the first unit function in Eq. (228) can be different from zero. The graph of $e(x,t_1)$ is shown in Fig. 67. The arrow indicates that as t_1

$$0 < t_1 < \frac{S}{v}$$
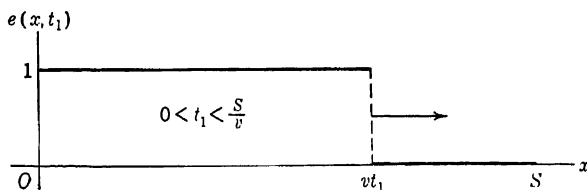
FIG. 67.
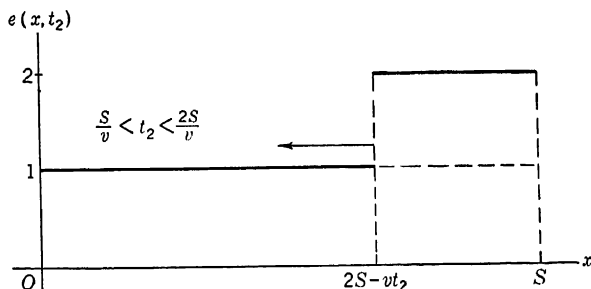
$$\frac{S}{v} < t_2 < \frac{2S}{v}$$

FIG. 68.

increases the wave moves toward the right. When t increases through S/v, we reach times t_2 such that $S/v < t_2 < 2S/v$. For any such time t_2 the first unit function in Eq. (228) is always 1, the second unit function may be different from zero, but all the unit functions which follow are zero. Hence the graph of $e(x,t_2)$ is as shown in Fig. 68. The arrow here indicates that as t_2 increases the second unit wave moves toward the left. When t increases through $2S/v$, we reach times t_3 such that $2S/v < t_3 < 3S/v$. For any such time t_3, the first unit function in Eq. (228) as well as

the second, is always 1.   The third unit function may be different
from zero, but all the unit functions which follow are zero.
Because of the minus sign, when the third function is 1 it cancels
the second, and the graph of $e(x,t_3)$ is as shown in Fig. 69.   When
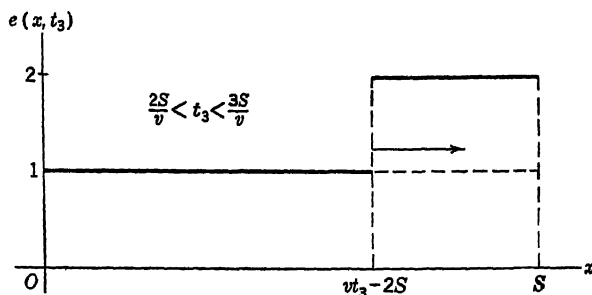
FIG. 69.

t increases through $3S/v$ we reach times t_4 such that

$$\frac{3S}{v} < t_4 < \frac{4S}{v}.$$

For any such time the first unit function in Eq. (228), as well as
the second and the third, is always 1.   The fourth may be zero,
and all which follow are zero.   Because of the two minus signs,
the third always cancels the second, and when the fourth is 1, it
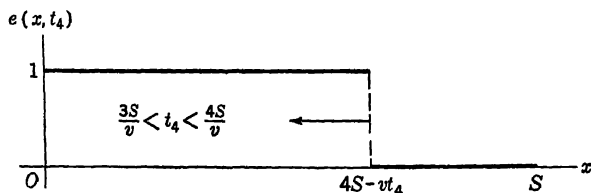
FIG. 70.

cancels the first.   Thus the graph of $e(x,t_4)$ is as shown in Fig. 70.
After this the cycle repeats, the graph for t_5 being similar to that
for t_1, the graph for t_6 being similar to that for t_2, etc., where
$(n-1)S/v < t_n < nS/v$.   The four steps of the cycle are (1) a

voltage wave which travels up the line with velocity v, Fig. 67; (2) a reflected wave from the open end of the line without change of sign, Fig. 68; (3) this reflected wave after reaching the closed end is reflected with a change of sign, Fig. 69; (4) after reaching the open end, it is again reflected retaining its minus sign, Fig. 70.

To find the current, we substitute the series from Eq. (224) in Eq. (221). The result is

$$I(x,p) = \frac{1}{Z_K}\frac{1}{p}(1 - \epsilon^{-2pS/v} + \cdots)[\epsilon^{-px/v} - \epsilon^{(x-2S)p/v}]$$

$$= \frac{1}{Z_K}\left\{\epsilon^{-(x/v)p}\frac{1}{p} - \epsilon^{-[(2S-x)/v]p}\frac{1}{p} - \epsilon^{-[(2S+x)/v]p}\frac{1}{p}\right.$$

$$\left. + \epsilon^{-[(4S-x)/v]p}\frac{1}{p} + \epsilon^{-[(4S+x)/v]p}\frac{1}{p} - \cdots \right\}. \quad (230)$$

Applying Eq. (227) with $b = x/v$, $(2S - x)/v$, etc., and item 1 we find

$$i(x,t) = \frac{1}{Z_K}\left[1\left(t - \frac{x}{v}\right) - 1\left(t - \frac{2S - x}{v}\right) - 1\left(t - \frac{2S + x}{v}\right)\right.$$

$$\left. + 1\left(t - \frac{4S - x}{v}\right) + 1\left(t - \frac{4S + x}{v}\right) - \cdots \right]. \quad (231)$$

This may be interpreted for times t_1, t_2, etc., by using considerations similar to those applied in the discussion of Eq. (228). We again find that the behavior for t_5 is like that for t_1. For the current, the four steps of the cycle are (1) a current wave of height
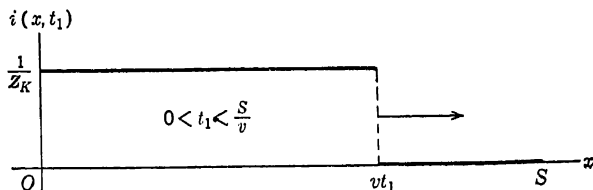
FIG. 71.

$1/Z_K$ which travels up the line with velocity v, Fig. 71; (2) this wave is reflected from the open end of the line with a change of

sign, Fig. 72; (3) this reflected wave, after reaching the closed end and completely canceling the first wave when $t = 2S/v$, is
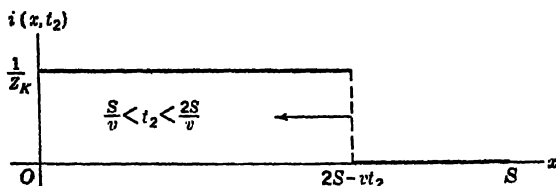
FIG. 72.
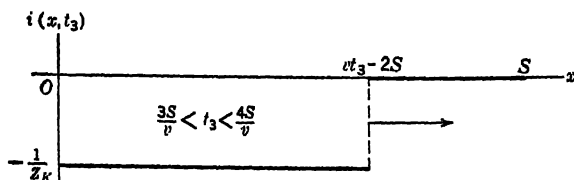
FIG. 73.
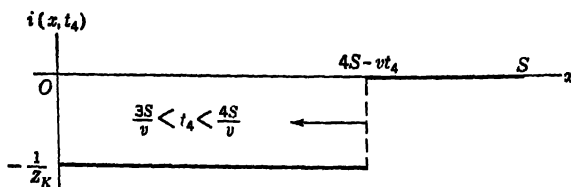
FIG. 74.

reflected without losing its minus sign, Fig. 73; (4) after reaching the open end, it is again reflected with a change of sign, Fig. 74.

61. References

A somewhat fuller, but elementary, account of Laplace transforms and a more extensive table than we have given here will be found in Churchill's *Modern Operational Mathematics in Engineering*.   This includes some theory and a number of applications to problems in heat conduction and in mechanical vibrations.

A summary of the transform method is given in Chap. 3, and approximate methods of finding direct and inverse transforms are

described in Chap. 11 of Brown and Campbell's *Principles of Servomechanisms*. Throughout this book Laplace transform methods are applied to problems arising in the analysis and synthesis of complex systems.

For a comprehensive treatment of the theory and application of the Laplace transform, the reader is referred to Gardner and Barnes's *Transients in Linear Systems*. Enough theory to serve as background for the discussion here given will be found in Chap. XIV of the author's *Treatise on Advanced Calculus*.

EXERCISE XXXI

1. Find the solution of $\dfrac{\partial f}{\partial t} = 3x + 6t$ which has $f(x,0) = 2 \cos x$.

2. Find the solution of $\dfrac{\partial f}{\partial t} = 4 \cos 2t$ which has $f(x,0) = \sin x$.

3. Find the solution of $\dfrac{\partial u}{\partial t} = e^{x+t}$ which has $f(x,0) = 3e^x$.

4. Find the solution of $\dfrac{\partial^2 f}{\partial x\, \partial t} = 4x - 4t$ which satisfies the boundary conditions $f(x,0) = 3 \sin x$ and $f(0,t) = \sin 2t$.

5. Find the solution of $\dfrac{\partial^2 u}{\partial x\, \partial t} = \sin x \sin t$ which satisfies the boundary conditions $f(x,0) = 2 \cos x$ and $f(0,t) = 2 \cos t$.

6. Show that in general the transforms of the second derivatives are given by $\text{Lap } \dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 F}{\partial x^2}$; $\text{Lap } \dfrac{\partial^2 f}{\partial x\, \partial t} = -f_x(x,0+) + p\,\dfrac{\partial F}{\partial x}$,

where $f_x(x,t) = \dfrac{\partial f}{\partial x}$; and

$$\text{Lap } \frac{\partial^2 f}{\partial t^2} = -f_t(x,0+) - pf(x,0+) + p^2 F(x,p),$$

where $f_t(x,t) = \dfrac{\partial f}{\partial t}$.

7. When a system whose equations contain all three partial derivatives is initially dead, $f(x,0+) = 0$, $f_x(x,0+) = 0$, and $f_t(x,0+) = 0$. Verify that in this case the relations of Prob. 6 reduce to those of Eq. (158).

8. Use the transform method as in Sec. 53 to show that the solution of $\frac{dy}{dt} = at$ which has $y(0+) = b$ is $y = \frac{1}{2}at^2 + b$.   Now replace y, t, a, b by F, x, $\left(\frac{8}{p^3} + \frac{2}{p}\right)$, $G(p)$, and thus check Eq. (165).

9. Use the transform method as in Sec. 53 to show that the solution of $\frac{dy}{dt} = ay$ which has $y(0+) = b$ is $y = be^{at}$.   Now replace y, t, a, b by U, x, $-2p$, $A(p)$, and thus check Eq. (171).

10. Use the transform method as in Sec. 53 to show that the solution of $\frac{d^2y}{dt^2} = a^2y$, when $a \neq 0$, which has $y(0+) = b$ and $y'(0) = c$ is $y = b \cosh at + \frac{c}{a}\sinh at = Ae^{-at} + Be^{at}$ if $b = A + B$ and $c = a(B - A)$.   Now replace y, t, e, a, A, B by E, x, ϵ, $\frac{p}{v}$, $A(p)$, $B(p)$, and thus check Eq. (184).

11. Find the solution of $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$ which satisfies the boundary conditions $u(x,0) = 0$ for $x > 0$ and $u(0,t) = \sin t$ for $t > 0$.

12. Find the solution of $\frac{\partial u}{\partial t} + 3\frac{\partial u}{\partial x} = 0$ which satisfies the boundary conditions $u(x,0) = 0$ for $x > 0$ and $u(0,t) = 5$ for $t > 0$.

13. Find the solution of $\frac{\partial u}{\partial t} + \frac{\partial u}{\partial t} = 0$ which satisfies the boundary conditions $u(x,0) = e^x$ for $x > 0$ and $u(0,t) = e^{-t}$ for $t > 0$.

14. Find the solution of $\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x} = 0$ which satisfies the boundary conditions $u(x,0) = 0$ for $x < 0$ and $u(0,t) = \sin t$ for $t > 0$.

15. Show the solution of $\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial x} = 0$, $v > 0$, which satisfies

the boundary conditions $u(x,0) = 0$ for $x > 0$ and $u(0,t) = g(t)$ for $t > 0$ is $u(x,t) = 0$ for $t < \frac{x}{v}$ and $u(x,t) = g\left(t - \frac{x}{v}\right)$ for $t > \frac{x}{v}$.

16. Check Prob. 14 by putting $x = -x_1$ in Prob. 11.

17. Find the solution of $\dfrac{\partial u}{\partial t} + \dfrac{\partial u}{\partial x} = 0$ which satisfies the boundary conditions $u(x,0) = 0$ for $x < 0$ and $u(0,t) = \sin t$ for $t < 0$. HINT: Put $t = -t_1$ and use Prob. 14.

18. Find the solution of $\dfrac{\partial u}{\partial t} + v\,\dfrac{\partial u}{\partial x} = -au$ which satisfies the boundary conditions $u(x,0) = 0$ for $x < 0$ and $u(0,t) = g(t)$ for $t > 0$.

The infinite transmission line of Fig. 75 has $L = 0.5$ henry per
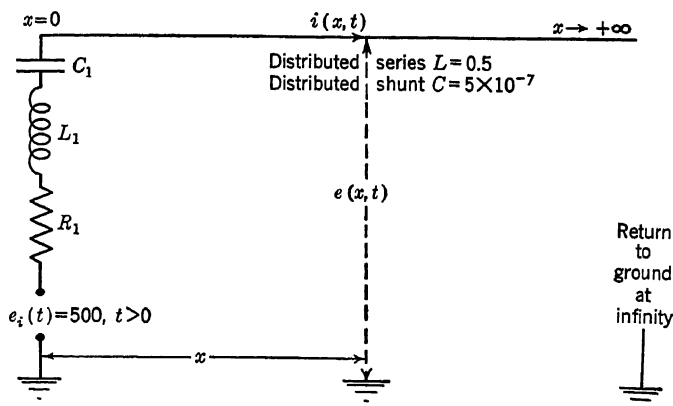
FIG. 75.   Infinite lossless transmission line.

mile and $C = 0.5$ microfarad per mile or 5×10^{-7} farad per mile. Hence $v = 1/\sqrt{LC} = 2,000$ and $Z_K = \sqrt{L/C} = 1,000$. At time $t = 0$ the line was dead, and the emf $e_i(t) = 500$ volts was suddenly impressed. Find the resulting potential $e(x,t)$ and current $i(x,t)$ if for the lumped parameters of the first element

19. $L_1 = 250$, $R_1 = 500$, no capacitance present.
20. $L_1 = 1,500$, $R_1 = 2,500$, no capacitance present.

21. $R_1 = 4,000$, $C_1 = 5 \times 10^{-5}$, no inductance present.

22. $L_1 = 5,000$, $R_1 = 9,000$, $C_1 = 10^{-4}$.

23. Find $e(x,t)$ and $e(0,t)$ for an infinite line with $v = 20$, $Z_K = 3$, in response to a suddenly impressed emf $e_i(t) = 540 \sin t$, if the initial element has $R_1 = 10$ in series with $L_1 = 1$.

Find $e(x,t)$ for an infinite line with constants v, Z_K in response to a suddenly impressed emf $e_i(t) = 1$ if the initial element in series with the line

24. Contains no impedance.   **25.** Is a resistance $R_1 = aZ_K$.

26. Is an inductance $L_1 = Z_K/a$.

27. Is a capacitance $C_1 = 1/aZ_K$.

An infinite line has constants v, Z_K. The suddenly impressed emf is $e_i(t)$. We define $g(z) = 0$ for $z < 0$ and $g(z) = e_i(z)$ for $z > 0$. Show that if the initial element in series with the line

28. Contains no impedance, $e(x,t) = g\left(t - \dfrac{x}{v}\right)$.

29. Is a resistance R_1, $e(x,t) = \dfrac{Z_K}{R_1 + Z_K} g\left(t - \dfrac{x}{v}\right)$.

30. The finite line of Fig. 76 is of length S and has constants
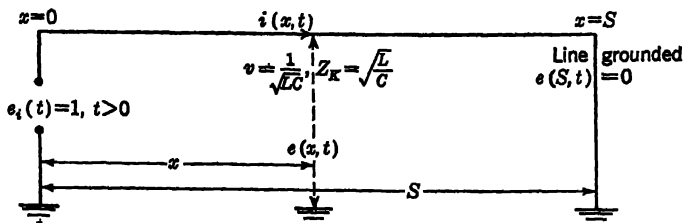
FIG. 76. Lossless transmission line of length S.

v, Z_K. There is no impedance in series with the line, and the suddenly impressed voltage at the home end is $e_i(t) = 1$, so that Eqs. (213) and (215) still hold and $1/p = A(p) + B(p)$. But at the far end the line is grounded, so that $e(S,t) = 0$. By putting

$x = S$ in Eq. (211) deduce that $0 = A(p)\epsilon^{-pS/v} + B(p)\epsilon^{pS/v}$. Solve the two equations as simultaneous in $A(p)$, $B(p)$. Use these values in Eqs. (211) and (212) and thus show that

$$\text{Lap } e(x,t) = E(x,p) = \frac{1}{p} \frac{1}{1 - \epsilon^{-2pS/v}} [\epsilon^{-px/v} - \epsilon^{(x-2S)p/v}],$$

$$\text{Lap } i(x,t) = I(x,p) = \frac{1}{Z_K} \frac{1}{p} \frac{1}{1 - \epsilon^{-2pS/v}} [\epsilon^{-px/v} + \epsilon^{(x-2S)p/v}].$$

31. Use the Maclaurin's series or infinite geometric series relation $\frac{1}{1-y} = 1 + y + y^2 + y^3 + \cdots$, $|y| < 1$, with $y = \epsilon^{-2pS/v}$, to derive the expansion

$$\frac{1}{1 - \epsilon^{-2pS/v}} = 1 + \epsilon^{-2pS/v} + \epsilon^{-4pS/v} + \epsilon^{-6pS/v} + \cdots.$$

32. Use the results of Probs. 30 and 31 to find the resulting potential $e(x,t)$.

33. Use the results of Probs. 30 and 31 to find the resulting current $i(x,t)$.

34. The finite line of Fig. 77 is of length S and has constants
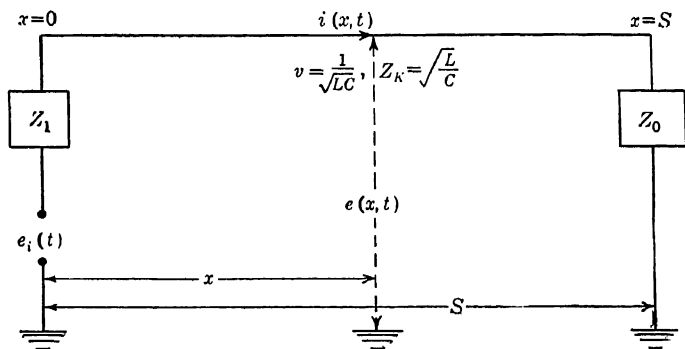
Fig. 77. Transmission line with initial impedance Z_1 and with output impedance Z_0.

v, Z_K. The suddenly impressed emf is $e_i(t)$. This acts at the home end through an initial impedance Z_1 in series with the line. At the far end the line is connected to the ground by a load or out-

put impedance Z_0. By applying reasoning like that used to derive Eq. (202) and the first part of Eq. (204) to the home end $x = 0$, deduce that $E_i(p) = E(0,p) + Z_1(p)I(0,p)$. And by applying similar reasoning to the far end $x = S$, deduce that $E(S,p) = Z_0(p)I(S,p)$.

35. Put $x = 0$ and $x = S$ in Eqs. (211) and (212). Substitute these results in the relations found in Prob. 34 and thus show that

$$Z_K E_i(p) = [Z_K + Z_1(p)]A(p) + [Z_K - Z_1(p)]B(p),$$
$$B(p) = A(p) \frac{Z_0(p) - Z_K}{Z_0(p) + Z_K} \epsilon^{-2pS/v}.$$

36. Show that if we introduce the abbreviations

$$M_{1K}(p) = \frac{Z_1(p) - Z_K}{Z_1(p) + Z_K} \quad \text{and} \quad M_{0K}(p) = \frac{Z_0(p) - Z_K}{Z_0(p) + Z_K}$$

the two equations of Prob. 35 may be written

$$\frac{Z_K E_i(p)}{Z_1(p) + Z_K} = A(p) - M_{1K}(p)B(p),$$

$$B(p) = A(p)M_{0K}(p)\epsilon^{-2pS/v}.$$

Solve these last two equations as simultaneous in $A(p)$, $B(p)$. Use the values found in Eqs. (211) and (212) and thus show that

$$E(x,p) = \frac{Z_K E_i(p)}{Z_1(p) + Z_K} \frac{\epsilon^{-px/v} + M_{0K}(p)\epsilon^{(x-2S)p/v}}{1 - M_{1K}(p)M_{0K}(p)\epsilon^{-2pS/v}},$$
$$I(x,p) = \frac{E_i(p)}{Z_1(p) + Z_K} \frac{\epsilon^{-px/v} - M_{0K}(p)\epsilon^{(x-2S)p/v}}{1 - M_{1K}(p)M_{0K}(p)\epsilon^{-2pS/v}}.$$

37. Observing that when $S \to +\infty$, $\epsilon^{-2pS/v} \to 0$, verify that in this case the relations of Prob. 36 reduce to those found for the infinite line by combining Eqs. (206), (207), and (203).

38. If Z_0 is a resistance numerically equal to Z_K, the output impedance is said to be matched to the characteristic impedance. Show that in this case $Z_0(p) = Z_K$, the output mismatch function $M_{0K}(p) = 0$, and the equations of Prob. 36 take the same form as those for the infinite line mentioned in Prob. 37.

39. If Z_1 is a resistance numerically equal to Z_K, the initial impedance is matched to the characteristic impedance. Show that in this case $Z_1(p) = Z_K$, the initial mismatch function

$M_{1K}(p) = 0$, and the equations of Prob. 36 reduce to

$$E(x,p) = \frac{1}{2}\left[E_i(p)\epsilon^{-px/v} + M_{0K}(p)\epsilon^{(x-2S)p/v}\right],$$

$$I(x,p) = \frac{1}{2Z_K}\left[E_i(p)\epsilon^{-px/v} - M_{0K}(p)\epsilon^{(x-2S)p/v}\right].$$

A finite line has constants v, Z_K and is of length S as in Fig. 77. The initial impedance Z_1 is matched to Z_K, $Z_1(p) = Z_K$, and the results of Prob. 39 hold. Find $e(x,t)$ in response to a suddenly impressed emf $e_i(t) = 1$ if at the far end

40. The line is open, $Z_0(p) = \infty$ so that $M_{0K}(p) = 1$.

41. The line is grounded, $Z_0(p) = 0$ so that $M_{0K} = -1$.

42. The load is a resistance $R_0 = aZ_K$ so that $Z_0(p) = aZ_K$ and

$$M_{0K}(p) = \frac{a-1}{a+1}.$$

43. Find $i(x,t)$ for the line of Prob. 40.

44. Find $i(x,t)$ for the line of Prob. 41.

45. Find $i(x,t)$ for the line of Prob. 42.

46. A finite line has constants v, Z_K and is of length S as in Fig. 77. The initial impedance is a resistance matched to Z_K, $R_1 = Z_K$. Hence $z_1(p) = Z_K$ and the results of Prob. 39 hold. The load at the far end is a resistance $R_0 = aZ_K$ so that

$$M_{0K}(p) = \frac{a-1}{a+1}.$$

The suddenly impressed emf is $e_i(t)$. We define $g(z) = 0$ for $z < 0$ and $g(z) = e_i(z)$ for $z > 0$. Show that

$$e(x,t) = \frac{1}{2}\left[g\left(t - \frac{x}{v}\right) + \frac{a-1}{a+1}g\left(t - \frac{2S-x}{v}\right)\right].$$

47. Use Prob. 46 to check Prob. 42.

48. Check Eqs. (220) and (221) by suitably specializing the relations of Prob. 36. Note that here $Z_0(p) = \infty$ so that $M_{0K}(p) = 1$, $Z_1(p) = 0$ so that $M_{1K}(p) = -1$, and $E_i(p) = 1/p$.

49. Check the relations of Prob. 30 by suitably specializing the relations of Prob. 36. Note that here $Z_0(p) = 0$ so that $M_{0K}(p) = -1$, $Z_1(p) = 0$ so that $M_{1K}(p) = -1$, and $E_i(p) = 1/p$.

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ANSWERS

Exercise I (Pages 5 to 6)

1. $2 + 29i$. 2. $5 + 5i$. 3. $-4i$. 4. 6.
5. $-13i$. 6. -13. 7. $52 - 78i$. 8. i.
9. $6 + 4i$. 10. $-4 - 6i$. 11. $5 - 12i$. 12. -676.
27. $x - (\sin 4x)/4$. 28. $(\sin 4x)/4 + 2 \sin 2x + 3x$.

29. $(\sin 4x)/4 - 2 \sin 2x + 3x$. 30. $\dfrac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}$,

$$\dfrac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}.$$

Exercise II (Pages 10 to 12)

1. $0.866 + 0.500i$. 2. -1. 3. -1.
4. 1. 5. 3.762. 6. $10.02i$.
7. 1.543. 8. $3.627i$. 9. $16.95 - 31.3i$.
10. $0.764 + 1.41i$. 11. $-8.42 - 54.0i$. 12. $-930 - 585i$.
13. $2.65 - 0.99i$. 14. $3.16 + 5.65i$. 15. $-1.54 - 9.94i$.
16. $-536 - 4010i$. 32. 1.3018.

Exercise III (Pages 18 to 19)

1. 7, 0. 2. $3, \pi/2 = 1.571$. 3. $5, -\pi/2 = -1.571$
4. 6.403, 2.246. 5. $9.220, -1.352$. 6. 8.062, 4.194.
9. $3.1416i$. 10. $1.5708i$. 11. $1.733 - 1.571i$.
12. $1.609 + 3.785i$. 13. $1.609 + 0.927i$. 14. $3.606 - 0.983i$.
15. -4. 16. $-10 - 3i$. 17. $-4 - 8i$.
18. $-18i$. 19. i. 20. 5.
21. $-1 + 1.732i$. 22. $4.242i$. 23. $12i$.
24. $90.63 + 42.26i$. 25. $1.813 + 0.845i$. 26. $-4i$.

27. $5i, -5i$. 28. $\pm(2.828 + 2.828i)$.
29. $\pm(3.01 + 0.166i)$. 30. $\pm(2.47 - 0.400i)$.
31. $-4, 2 \pm 3.464i$. 32. $\pm 2, \pm 2i$.
33. $2i, \pm 0.951 + 0.309i$, 34. $1, (0.5 + 0.866i)$,
$\pm 0.588 + 0.809i$. $(0.5 - 0.866i)$.

Exercise IV (Pages 26 to 29)

32. $y = 6.16$ ft., $H = 4.08$ lb., $T = 4.20$ lb.
33. $H = 22.7$ tons, $T = 26.6$ tons, $s = 3180$ ft., weight $= 28$ tons.

Exercise V (Pages 42 to 47)

1. $i = -0.849e^{-0.15t} \sin 141.4t.$ **3.** $0.171 \sin (120\pi t + 84.1°).$

4. $0.171 \cos (120\pi t + 84.1°).$ **5.** $0.0163 \sin (360\pi t - 44.4°).$

6. $0.00075 \sin (600\pi t - 107.5°).$

7. $0.086 \sin (120\pi t + 84.1°) + 0.032 \sin (360\pi t - 44.4°).$

8. $0.086 \sin (120\pi t + 84.1°) + 0.0015 \sin (600\pi t - 107.5°)$

In Probs. 9 to 12 let $X = L\omega - \dfrac{1}{\omega C}$, $|Z| = \sqrt{R^2 + X^2}$, $\theta_Z = \tan^{-1}\dfrac{X}{R}.$

9. $\dfrac{1}{|Z|} \sin (\omega t - \theta_Z).$

10. $\dfrac{1}{|Z|} \cos (\omega t - \theta_Z).$

11. $\dfrac{A}{|Z|} \cos (\omega t - \theta_Z) + \dfrac{B}{|Z|} \sin (\omega t - \theta_Z).$

12. $\dfrac{1}{|Z|} \cos (\omega t + \alpha - \theta_Z).$

14. $s = 4.85 \sin (4t - 86.0°)$

15. $\theta = 0.000213 \sin (300t - 179.4°).$

16. $i_1 + i_2 + i_3 = 0, \ Z_1 i_1 - Z_2 i_2 = e_1 - e_2, \ Z_2 i_2 - Z_3 i_3 = e_2 - e_3.$

18. $i_1 + i_2 - i_5 = 0, i_5 + i_3 + i_4 = 0, -i_1 - i_2 + i_6 = 0, i_1 Z_1 - i_2 Z_2 = 0,$
$i_2 Z_2 + i_5 Z_5 - i_3 Z_3 + i_6 Z_6 = e_6, \ i_3 Z_3 - i_4 Z_4 = 0.$

19. $i_6 = \dfrac{e_6}{Z},$ where $Z = \dfrac{Z_1 Z_2}{Z_1 + Z_2} + \dfrac{Z_3 Z_4}{Z_3 + Z_4} + Z_5 + Z_6.$

21. $i_1 = P/Q \ \sin \ (\omega t + p - q),$ where $P\epsilon^{ip} = R_3 e_1 + (L_3 e_1 - M e_3)\omega j,$
and $Q\epsilon^{iq} = M^2\omega^2 + R_1 R_3 - L_1 L_3 \omega^2 + (L_1 R_3 + L_3 R_1)\omega j.$

Exercise VI (Pages 54 to 57)

1. $-6.$ **2.** $106.$ **3.** $e - e^{-3}.$

4. $\frac{5}{2}(3 \ln 3 - 2).$ **5.** $(\frac{4}{9})^{1/2}.$ **6.** $[\frac{3}{8}(e^{12} - e^4)]^{1/2}.$

7. $(\frac{1}{5})^{1/2}.$ **8.** $(6,348)^{1/2}.$ **9.** $5, 5.$

10. $\frac{8}{3}, (6\frac{4}{5})^{1/2}.$ **11.** $48\frac{4}{5}, 16.$

12. $(\sin 2)/2, (2 + \sin 2 \cos 2)^{1/2}/2.$

13. $4, (6\frac{4}{3})^{1/2}.$

14. $16, (4,048/7)^{1/2}.$

15. $(1 - \cos 2)/2, (2 - \sin 2 \cos 2)^{1/2}/2.$

16. $(\sin^2 2)/4, (4 - \sin 4 \cos 4)^{1/2}/4.$

17. $0.$ **18.** $4.$ **19.** $2.$

20. $-5.$ **21.** $2, (1\frac{6}{3})^{1/2}.$ **22.** $2E/\pi, E/2^{1/2}.$

23. $E/\pi, E/2.$

Exercise VII (Pages 63 to 66)

2. $\frac{2}{5}.$ **3.** $\frac{1}{10}.$ **4.** $\frac{1}{2}.$ **5.** $\frac{2}{3}.$

6. $12.$ **7.** $30.$ **13.** $(25,450)^{1/2}.$ **14.** $(16,288)^{1/2}.$

15. $(29{,}952)^{1/2}$.   **16.** $(3.84)^{1/2}$.   **17.** $(32{,}500)^{1/4}$.   **18.** $(204.5)^{1/2}$.
19. $(82{,}450)^{1/4}$.   **20.** $(3.29)^{1/2}$.   **21.** $(325)^{1/2}$.   **22.** $(325)^{1/2}$.
23. $20{,}360$.   **24.** 282.   **25.** 494.   **26.** 323.
27. 325.

Exercise VIII (Pages 71 to 73)

1. $\Sigma(-1)^{n+1}2n^{-1}\sin nx$.

2. $\pi^2/3 + \Sigma(-1)^n 4n^{-2}\cos nx$.

3. $\Sigma(-1)^{n+1}2(\pi^2 n^{-1} - 6n^{-3})\sin nx$.

4. $\dfrac{\sinh \pi}{\pi}\left[1 + \sum(-1)^n 2\,\dfrac{\cos nx - n\sin nx}{1 + n^2}\right]$.

5. $1 - \dfrac{\cos x}{2} + \sum(-1)^n\dfrac{2\cos(n+1)x}{n^2 + 2n}$.

6. $-\dfrac{\sin x}{2} + \sum(-1)^{n+1}\dfrac{2n+2}{n^2+2n}\sin(n+1)x$.

7. $\pi/2 - \Sigma 2m^{-1}\sin mx, m$ odd.

8. $-\dfrac{\pi}{4} + \sum\dfrac{2\cos(2n-1)x}{\pi(2n-1)^2} + \sum(-1)^{n+1}\dfrac{\sin nx}{n}$.

9. $\pi - \Sigma 2n^{-1}\sin nx$.

10. $\tfrac{1}{3}\pi^2 + \Sigma 4n^{-2}\cos nx - 4\pi n^{-1}\sin nx$.

11. $2\pi^3 + \Sigma 12\pi n^{-1}\cos nx + \Sigma(12n^{-3} - 8\pi^2 n^{-1})\sin nx$.

12. $\dfrac{e^{2\pi} - 1}{\pi}\left(\dfrac{1}{2} + \sum\dfrac{\cos nx - n\sin nx}{1 + n^2}\right)$.

13. $\pi\sin x - 1 - \dfrac{\cos x}{2} + \sum\dfrac{2\cos(n+1)x}{n^2 + 2n}$.

14. $\pi\cos x - \dfrac{\sin x}{2} - \sum\dfrac{2n+2}{n^2+2n}\sin(n+1)x$.

15. $\pi/2 + \Sigma 2m^{-1}\sin mx$, m odd.

16. $\dfrac{\pi}{4} - \dfrac{2\cos nx}{n^2} + \sum(-1)^{n+1}\dfrac{\sin nx}{n}$.

17. $5 + \sum\dfrac{20}{\pi}\sin\dfrac{m\pi x}{10}$, m odd.

18. $\dfrac{5}{4} + \sum\left\{\dfrac{5[(-1)^n - 1]}{n^2\pi^2}\cos\dfrac{n\pi x}{5} + \dfrac{5(-1)^{n+1}}{n\pi}\sin\dfrac{n\pi x}{5}\right\}$.

19. $\dfrac{e^5 - 1}{10} + \sum\left\{\dfrac{5[(-1)^n e^5 - 1]}{5^2 + n^2\pi^2}\cos\dfrac{n\pi x}{5} + \dfrac{n\pi[(-1)^{n+1}e^5 + 1]}{5^2 + n^2\pi^2}\sin\dfrac{n\pi x}{5}\right\}$.

20. $\dfrac{1}{\pi} + \dfrac{1}{2}\sin\dfrac{\pi x}{5} - \dfrac{2}{\pi}\sum\dfrac{\cos\dfrac{2n\pi x}{5}}{4n^2 - 1}$.

21. $\dfrac{25}{6} + \sum\left\{\dfrac{50(-1)^n}{n^2\pi^2}\cos\dfrac{n\pi x}{5} + \left[\dfrac{25(-1)^{n+1}}{n\pi} + \dfrac{50[(-1)^n - 1]}{n^3\pi^3}\right]\sin\dfrac{n\pi x}{5}\right\}$.

Exercise IX (Pages 80 to 83)

1. $\dfrac{2}{\pi} - \sum \dfrac{4 \cos 2nx}{\pi(4n^3 - 1)}$.

2. $\dfrac{2}{\pi} + \sum (-1)^{n+1} \dfrac{4 \cos 2nx}{\pi(4n^3 - 1)}$.

3. $\frac{1}{2} - (\cos 2x)/2$.

4. $\frac{1}{2} + (\cos 2x)/2$.

5. $\dfrac{10}{3\pi} \sin x + \sum \dfrac{2}{\pi} \left[\dfrac{1}{4(n + 1)^3 - 1} - \dfrac{1}{4n^3 - 1} \right] \sin (2n + 1)x$.

6. $\dfrac{2}{3\pi} \sin x + \sum (-1)^n \dfrac{2}{\pi} \left[\dfrac{1}{4(n + 1)^3 - 1} + \dfrac{1}{4n^3 - 1} \right] \sin (2n + 1)x$.

7. $\frac{3}{4} \sin x - \frac{1}{4} \sin 3x$.

8. $\frac{1}{4} \sin x + \frac{1}{4} \sin 3x$.

9. $\dfrac{2}{3\pi} \cos x - \sum \dfrac{2}{\pi} \left[\dfrac{1}{4(n + 1)^3 - 1} + \dfrac{1}{4n^3 - 1} \right] \cos (2n + 1)x$.

10. $\dfrac{10}{3\pi} \cos x + \sum (-1)^n \dfrac{2}{\pi} \left[\dfrac{1}{4(n + 1)^3 - 1} - \dfrac{1}{4n^3 - 1} \right] \cos (2n + 1)x$.

11. $\frac{1}{4} \cos x - \frac{1}{4} \cos 3x$.

12. $\frac{3}{4} \cos x + \frac{1}{4} \cos 3x$.

13. $2 - \sum \dfrac{16}{m^2\pi^2} \cos \dfrac{m\pi x}{4}$, m odd.

14. $\sum \dfrac{16}{m^2\pi^2} \cos \dfrac{m\pi x}{4}$, m odd.

15. $2 + \sum \dfrac{8}{\pi} (-1)^n \dfrac{1}{2n - 1} \cos \dfrac{(2n - 1)\pi x}{4}$.

16. $\dfrac{3}{2} + \sum \dfrac{8}{n^2\pi^2} \left(\cos \dfrac{n\pi}{2} - 1 \right) \cos \dfrac{n\pi x}{4}$.

17. $\sum \dfrac{8}{\pi} (-1)^{n+1} n^{-1} \sin \dfrac{n\pi x}{4}$.

18. $\sum \dfrac{4}{n\pi} \sin \dfrac{n\pi x}{2}$.

19. $\sum \dfrac{8}{n\pi} \left(\cos \dfrac{n\pi}{2} - \cos n\pi \right) \sin \dfrac{n\pi x}{4}$.

20. $\sum \left[\dfrac{8}{n^2\pi^2} \sin \dfrac{n\pi}{2} - \dfrac{4}{n\pi} \cos n\pi \right] \sin \dfrac{n\pi x}{4}$.

Exercise X (Pages 87 to 89)

8. $B_1 = 0.373$, $B_2 = -0.173$, $B_3 = 0.100$, $B_4 = -0.057$, $B_5 = 0.027$.

9. $B_1 = 1.24$, $B_2 = 0$, $B_3 = 0.33$, $B_4 = 0$, $B_5 = 0.09$.

10. $B_1 = 0.496$, $B_2 = 0$, $B_3 = 0.066$, $B_4 = 0$, $B_5 = 0.036$.

11. $B_1 = 8$, $B_2 = 0$, $B_3 = -2$, $B_4 = 0$, $B_5 = 0$.

Exercise XII (Pages 103 to 106)

2. 6×10^7 cal./day, 14 kg./day.

3. 1.3×10^6 cal./day.

4. 15.3 kg./day.

5. 3 kwh./day.

10. 1.38×10^9 cal./day.

14. 23.7 cal./sec.

15. 392 cal./sec.

16. 393 cal./sec.

17. 4760 cal./sec.

18. 4760 cal./sec.

Exercise XIII (Pages 108 to 110)

1. $z = f(y)$.

2. $z = 2x^2y + f(y)$.

3. $z = f(x)$.

4. $z = 2y^2 + f(x)$.

5. $z = x^3 + 3xy^2 + f(y)$.

6. $z = x \sin \dfrac{y}{x} + f(x)$.

7. $z = f(x) + g(y)$.

8. $z = x^2y + 2xy^2 + f(x) + g(y)$.

9. $z = -\tfrac{1}{2}e^{2x-y} + f(x) + g(y)$.

10. $u = x^2 \ln t + f(x) + g(t)$.

11. $u = 2x^3t^2 + f(x) + tg(x)$.

12. $u = 4te^{2x} + f(t) + xg(t)$.

13. $u = -\tfrac{1}{4} \sin (2x - 3y) + f(y) + xg(y)$.

14. $u = 2xy^3 + f(x) + yg(x)$.

15. $u = 3p^2x^2 + f(x) + g(p)$.

16. $\ln z = x^2y + f(y)$.

17. $z^2 = x^2 - 2xy + g(y)$.

18. $e^{-z} = -xy + f(y)$.

19. $u = -2x/y - 2/y^2 + f(y)e^{xy}$.

20. $u = -2t + e^x f(t)$.

21. $u = f(p)e^{-2px}$.

22. $u = f(p)e^{-(4+2p)x}$.

23. $u = f(y)e^{2xy} + g(y)e^{-2xy}$.

24. $u = f(t) \sin 2tx - g(t) \cos 2tx$.

25. $u = f(p)e^{px/v} + g(p)e^{-px/v}$.

26. $u = f(p) \sin \dfrac{px}{v} + g(p) \cos \dfrac{px}{v}$.

27. $z = x^3y^2 + y^{-1}f(x) + g(y)$.

Exercise XIV (Pages 112 to 114)

1. $3\dfrac{\partial z}{\partial x} + 2\dfrac{\partial z}{\partial y} = 0$.

2. $\dfrac{\partial z}{\partial y} = 0$.

3. $\dfrac{\partial z}{\partial x} = 2y$.

4. $x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y} = 0$.

5. $2\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

6. $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

7. $2\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x\, \partial y} - 3\dfrac{\partial^2 z}{\partial y^2} = 0$.

8. $\dfrac{\partial^2 z}{\partial x\, \partial y} = \dfrac{1}{x}$.

9. $\dfrac{\partial^2 z}{\partial y^2} = 2$.

10. $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x\, \partial y} = 0$.

11. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2} = 0$.

12. $z\dfrac{\partial^2 z}{\partial x\, \partial y} - \dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y} = 0$.

Exercise XV (Pages 118 to 119)

1. $z = f(5x + y) + g(x + y)$.

2. $z = f(y) + g(2x + y)$.

3. $z = f(x - y) + g(3x - 2y)$.

4. $z = f(x) + g(x - 3y)$.

5. $z = f(3x + y) + xy(3x + y)$.

6. $z = f(2x + 3iy) + g(2x - 3iy)$.

7. $z = -2y^3/3 + f(x) + g(4x + y)$.

8. $z = x^4 + f(2x + y) + g(2x - y)$.

Exercise XVI (Pages 124 to 127)

2. $z = ce^{ax}e^{2ay}$.

3. $z = cx^ay^a$.

4. $z = cx^ay^{-a}$.

5. $z = ce^{kx}e^{ky}$.

6. $z = ce^{kx}e^{(2-k)y}$.

7. $z = ce^{kx}e^{ky}$.

8. $z = (c_1 e^{2ax} + c_2 e^{-2ax})(c_3 e^{ay} + c_4 e^{-ay})$,

   $z = (c_5 + c_6 x)(c_7 + c_8 y)$,

   $z = (c_9 \sin 2bx + c_{10} \cos 2bx)(c_{11} \sin by + c_{12} \cos by)$.

9. $z = e^{2ax}(c_1 e^{ay} + c_2)$,

   $z = c_3 y + c_4$.

10. $z = (c_1 e^{2ax} + c_2 e^{-2ax})(c_3 \sin ay + c_4 \cos ay)$,

    $z = (c_5 + c_6 x)(c_7 + c_8 y)$,

    $z = (c_9 \sin 2bx + c_{10} \cos 2bx)(c_{11} e^{by} + c_{12} e^{-by})$.

11. $U = (c_1 e^{3ax} + c_2 e^{-3ax})(c_3 e^{at} + c_4 e^{-at})$,

    $U = (c_5 + c_6 x)(c_7 + c_8 t)$,

    $U = (c_9 \sin 3bx + c_{10} \cos 3bx)(c_{11} \sin bt + c_{12} \cos bt)$.

12. $U = (c_1 e^{3ax} + c_2 e^{-3ax})(c_3 \sin at + c_4 \cos at)$,

    $U = (c_5 + c_6 x)(c_7 + c_8 t)$,

    $U = (c_9 \sin 3bx + c_{10} \cos 3bx)(c_{11} e^{bt} + c_{12} e^{-bt})$.

13. $U = e^{at}(c_1 e^{3ax} + c_2)$,

    $U = c_3 x + c_4$.

Exercise XIX (Pages 143 to 146)

1. $\dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} = \sigma E_z + K \dfrac{\partial E_z}{\partial t}$.

Exercise XX (Pages 152 to 155)

1. $\sum \dfrac{200}{m\pi} \sin \dfrac{m\pi x}{30} e^{-m\pi y/30}$, m odd.　　2. $\sum (-1)^{n+1} \dfrac{48}{n\pi} \sin \dfrac{n\pi x}{6} e^{-n\pi y/6}$.

3. $-\sum \dfrac{8}{n\pi} \sin \dfrac{n\pi x}{2} e^{-n\pi y/2}$.　　　　4. $100 \sin \dfrac{\pi x}{40} e^{-\pi y/40}$.

5. $4 \sin \dfrac{\pi x}{3} e^{-\pi y/3} - 6 \sin \dfrac{\pi x}{5} e^{-\pi y/5}$.

6. $\sum \dfrac{20}{n\pi} \left(1 - \cos \dfrac{n\pi}{2}\right) \sin \dfrac{n\pi x}{8} e^{-n\pi y/8}$.

7. $\sum \dfrac{4c}{m\pi} \sin \dfrac{m\pi x}{L} e^{-m\pi y/L}$, m odd.

8. $\sum \dfrac{2cL}{n\pi} (-1)^{n+1} \sin \dfrac{n\pi x}{L} e^{-n\pi y/L}$.

9. $-\sum \dfrac{2cL}{n\pi} \sin \dfrac{2n\pi x}{L} e^{-2n\pi y/L}$.

Exercise XXI (Pages 158 to 160)

1. $\dfrac{1}{2} + \sum \dfrac{2}{m\pi} \left(\dfrac{r}{9}\right)^m \sin m\theta$, m odd.　　2. $4r \sin \theta + 2r^2 \sin 2\theta$.

3. $2\pi - \Sigma 4n^{-1}r^n \sin n\theta$.　　　　4. $3r^3 \cos (3\theta - 25°)$.

6. $\sum \dfrac{400}{m\pi} \left(\dfrac{r}{a}\right)^{3m} \sin 3m\theta$, m odd.

7. $(4r - 4r^{-1}) \cos \theta + (4r + 4r^{-1}) \sin \theta$.

Exercise XXII (Pages 166 to 170)

1. $\sum (-1)^{n+1} \dfrac{200}{n\pi} \sin \dfrac{n\pi x}{50} e^{-bn^2 t}$, where $b = a^2\pi^2/2{,}500$.

2. $100 - \sum \dfrac{200}{n\pi} \sin \dfrac{n\pi x}{50} e^{-bn^2 t}$, where $b = a^2\pi^2/2{,}500$.

3. $x + \sum (-1)^{n+1} \dfrac{100}{n\pi} \sin \dfrac{n\pi x}{50} e^{-bn^2 t}$, where $b = a^2\pi^2/2{,}500$.

4. $50 - x + \sum \dfrac{100}{n\pi} k_n \sin \dfrac{n\pi x}{50} e^{-bn^2 t}$, where $b = a^2\pi^2/2{,}500$, $k_n = 1$ for n odd
   and $k_n = -3$ for n even.

5. $25 + x - \sum \dfrac{50}{n\pi} \sin \dfrac{n\pi x}{25} e^{-bn^2 t}$, where $b = a^2\pi^2/625$.

6. $50 + 2x - \sum \dfrac{200}{m\pi} \sin \dfrac{m\pi x}{50} e^{-bm^2 t}$, m odd, where $b = a^2\pi^2/2{,}500$.

7. $37.5°$.

8. $\sum \dfrac{100}{n\pi} \left(\cos \dfrac{n\pi}{2} - \cos n\pi\right) \sin \dfrac{n\pi x}{80} e^{-bn^2 t}$, where $b = a^2\pi^2/6{,}400$.

9. $8.0°$.   10. $74.9°$.   11. $42.7°$.   12. $37.5°$.

13. $30.5°$.   14. $25°$.   15. 50.

17. $90 - \sum \dfrac{720}{m^2\pi^2} \cos \dfrac{m\pi x}{60} e^{-bm^2 t}$, m odd, where $b = \pi^2/1{,}800$.

18. $x^2 - 120x + 2{,}400 - \sum \dfrac{14{,}400}{n^2\pi^2} \cos \dfrac{n\pi x}{60} e^{-bn^2 t}$, where $b = \pi^2/1{,}800$.

20. $\sum \dfrac{800}{m^2\pi^2} (-1)^{(m-1)/2} \sin \dfrac{m\pi x}{40} e^{-bm^2 t}$, m odd, where $b = a^2\pi^2/1{,}600$.

21. See answer to Prob. 20.

24. $\sum (-1)^{n+1} \dfrac{20}{n\pi} \sin \dfrac{n\pi x}{10} e^{-c_n t}$, where $c_n = \dfrac{n^2\pi^2}{50} + 5$.   **32.** $80.8°$.

Exercise XXIII (Pages 175 to 176)

1. $e = E \sin \dfrac{x}{500} \epsilon^{-4t}$,

   $i = -\dfrac{E}{1{,}500} \cos \dfrac{x}{500} \epsilon^{-4t}$.

2. $e = E_1 \sin \dfrac{x}{1{,}000} \epsilon^{-t} + E_{10} \sin \dfrac{x}{100} \epsilon^{-100t}$,

   $i = -\dfrac{E_1}{3{,}000} \cos \dfrac{x}{1{,}000} \epsilon^{-t} - \dfrac{E_{10}}{300} \cos \dfrac{x}{100} \epsilon^{-100t}$.

3. $e = \sum \dfrac{4E}{m\pi} \sin \dfrac{mx}{1,000}\, \epsilon^{-m^2t},$

$\quad i = -\sum \dfrac{4E}{3,000\pi} \cos \dfrac{mx}{1,000}\, \epsilon^{-m^2t},\ m$ odd.

4. $e = \sum \dfrac{2E}{n\pi}\,(-1)^{n+1} \sin \dfrac{nx}{1,000}\, \epsilon^{-n^2t},$

$\quad i = \sum \dfrac{2E}{3,000\pi}\,(-1)^n \cos \dfrac{nx}{1,000}\, \epsilon^{-n^2t}.$

5. $e = \sum \dfrac{2E}{n\pi}\,(-1)^{n+1} \sin \dfrac{n\pi x}{S}\, \epsilon^{-bn^2t},$

$\quad i = \sum \dfrac{2E}{RS}\,(-1)^n \cos \dfrac{n\pi x}{S}\, \epsilon^{-bn^2t},$ where $b = \dfrac{\pi^2}{RCS^2}.$

6. $e = \sum \dfrac{4E}{m\pi} \sin \dfrac{m\pi x}{S}\, \epsilon^{-bm^2t},$

$\quad i = -\sum \dfrac{4E}{RS} \cos \dfrac{m\pi x}{S}\, \epsilon^{-bm^2t},\ m$ odd, where $b = \dfrac{\pi^2}{RCS^2}.$

7. $e = \dfrac{Ex}{S} + \sum \dfrac{2E}{n\pi}\,(-1)^n \sin \dfrac{n\pi x}{S}\, \epsilon^{-bn^2t},$

$\quad i = -\dfrac{E}{RS} + \sum \dfrac{2E}{RS}\,(-1)^{n+1} \cos \dfrac{n\pi x}{S}\, \epsilon^{-bn^2t},$ where $b = \dfrac{\pi^2}{RCS^2}.$

Exercise XXIV (Pages 180 to 183)

1. $2 \sin \dfrac{\pi x}{S} \cos \dfrac{\pi vt}{S}.$

2. $\dfrac{p}{4}\left(3 \sin \dfrac{\pi x}{S} \cos \dfrac{\pi vt}{S} - \sin \dfrac{3\pi x}{S} \cos \dfrac{3\pi vt}{S}\right).$

3. $\sum \dfrac{8pS^2}{m^3\pi^3} \sin \dfrac{m\pi x}{S} \cos \dfrac{m\pi vt}{S},\quad m$ odd.

4. $p \sin \dfrac{k\pi x}{S} \cos \dfrac{k\pi vt}{S}.$

5. $\sum \dfrac{8p}{m^2\pi^2}\,(-1)^{(m-1)/2} \sin \dfrac{m\pi x}{S} \cos \dfrac{m\pi vt}{S},\quad m$ odd.

6. $\dfrac{5S}{3\pi v} \sin \dfrac{3\pi x}{S} \sin \dfrac{3\pi vt}{S}.$

7. $\dfrac{qS}{12\pi v}\left(9 \sin \dfrac{\pi x}{S} \sin \dfrac{\pi vt}{S} - \sin \dfrac{3\pi x}{S} \sin \dfrac{3\pi vt}{S}\right).$

8. $\sum \dfrac{8qS^3}{m^4\pi^4 v} \sin \dfrac{m\pi x}{S} \sin \dfrac{m\pi vt}{S},\quad m$ odd.

9. $\dfrac{qS}{n\pi v} \sin \dfrac{k\pi x}{S} \sin \dfrac{k\pi vt}{S}.$

10. $\sum \dfrac{4qS}{n^2\pi^2 v} \sin \dfrac{n\pi}{2} \sin \dfrac{wn\pi}{2S} \sin \dfrac{n\pi x}{S} \sin \dfrac{n\pi vt}{S}.$

11. $3 \sin \dfrac{2\pi x}{S} \cos \dfrac{2\pi vt}{S} + \dfrac{4S}{5\pi} \sin \dfrac{5\pi x}{S} \sin \dfrac{5\pi vt}{S}.$

Exercise XXV (Pages 188 to 190)

1. $e = E \sin \dfrac{\pi x}{S} \cos \dfrac{\pi v t}{S}$,

$\quad i = I_0 - \dfrac{E}{vL} \cos \dfrac{\pi x}{S} \sin \dfrac{\pi v t}{S}$.

2. $e = E_2 \sin \dfrac{2\pi x}{S} \cos \dfrac{2\pi v t}{S} + E_5 \sin \dfrac{5\pi x}{S} \cos \dfrac{5\pi v t}{S}$,

$\quad i = I_0 - \dfrac{E_2}{vL} \cos \dfrac{2\pi x}{S} \sin \dfrac{2\pi v t}{S} - \dfrac{E_5}{vL} \cos \dfrac{5\pi x}{S} \sin \dfrac{5\pi v t}{S}$.

3. $e = \sum \dfrac{4E}{m\pi} \sin \dfrac{m\pi x}{S} \cos \dfrac{m\pi v t}{S}$,

$\quad i = I_0 - \dfrac{1}{vL} \sum \dfrac{4E}{m\pi} \cos \dfrac{m\pi x}{S} \sin \dfrac{m\pi v t}{S}$, m odd.

4. $e = \sum \dfrac{2E}{n\pi} (-1)^{n+1} \sin \dfrac{n\pi x}{S} \cos \dfrac{n\pi v t}{S}$,

$\quad i = I_0 + \dfrac{1}{vL} \sum \dfrac{2E}{n\pi} (-1)^n \cos \dfrac{n\pi x}{S} \sin \dfrac{n\pi v t}{S}$.

5. $e = \dfrac{Ex}{S} + \sum \dfrac{2E}{n\pi} (-1)^n \sin \dfrac{n\pi x}{S} \cos \dfrac{n\pi v t}{S}$,

$\quad i = - \dfrac{Et}{LS} + \dfrac{1}{vL} \sum \dfrac{2E}{n\pi} (-1)^{n+1} \cos \dfrac{n\pi x}{S} \sin \dfrac{n\pi v t}{S}$.

7. $e = \sum \dfrac{4E}{m^2 \pi^2} (-1)^{(m-1)/2} \sin \dfrac{m\pi x}{S} \cos \dfrac{m\pi v t}{S}$, m odd.

8. $e = E - \sum \dfrac{4E}{m\pi} \sin \dfrac{m\pi x}{S} \cos \dfrac{m\pi v t}{S}$,

$\quad i = \dfrac{1}{vL} \sum \dfrac{4E}{m\pi} \cos \dfrac{m\pi x}{S} \sin \dfrac{m\pi v t}{S}$, m odd.

13. $e = \dfrac{6{,}000 n \pi I_0 [1 - (-1)^n \epsilon^{0.002}]}{n^2 \pi^2 + 4 \times 10^{-6}} \sin \dfrac{n\pi x}{50}\, \epsilon^{-30.3t} \left(\dfrac{30.3}{\beta_n} \sin \beta_n t + \cos \beta_n t \right)$.

14. $e = \dfrac{E \sinh 0.00004x}{\sinh 0.002}$

$\qquad\qquad + \sum \dfrac{2E n \pi (-1)^n}{n^2 \pi^2 + 4 \times 10^{-6}} \sin \dfrac{n\pi x}{50}\, \epsilon^{-30.3t} \left(\dfrac{30.3}{\beta_n} \sin \beta_n t + \cos \beta_n t \right)$.

Exercise XXVIII (Pages 213 to 217)

1. $5/p$.

2. $5/p^2$.

3. $\dfrac{4p - 2}{p^2}$.

4. $\dfrac{5}{p + 1}$.

5. $\dfrac{2}{(p + 3)^2}$.

6. $\dfrac{2p}{(p + 3)^2}$.

7. $\dfrac{20}{p^2 - 25}$.

8. $\dfrac{4p}{p^2 - 25}$.

9. $\dfrac{16}{p(p^2 - 16)}$.

10. $\dfrac{15}{p^2 + 9}$.

11. $\dfrac{5p}{p^2 + 9}$.

12. $\dfrac{-25}{p(p^2 + 25)}$.

13. $\dfrac{6p - 48}{p^5}$.

14. $\dfrac{1}{(p + 2)^2 + 1}$.

15. $\dfrac{p + 3}{(p + 3)^2 + 4}$.

16. $\dfrac{6e^{-4p}}{p}$.

17. $\dfrac{2e^{-4p}}{p^2}$.

18. $\dfrac{8p + 2}{p^2} e^{-4p}$.

19. $\dfrac{e^{-2p}}{p + 1}$.

20. $\dfrac{e^{-2}e^{-2p}}{p + 1}$.

21. $\dfrac{-e^{-\pi p}}{p^2 + 1}$.

22. 3.

23. $7t$.

24. $t^2 + 6t$.

25. $4e^{-5t}$.

26. $2te^{-3t}$.

27. $2 - 2e^{-3t}$.

28. $2 \sin 2t$.

29. $4 \cos 2t$.

30. $(2 - 6t)e^{-3t}$.

31. $2 \sinh 2t$.

32. $4 \cosh 2t$.

33. $2 - 2 \cos t$.

34. 0 for $t < \pi$, $-\sin t$ for $t > \pi$.

35. $2e^{-2t} \sin 3t$.

36. $6e^{-2t} \cos 3t$.

Exercise XXIX (Pages 229 to 232)

1. $x = \cos 2t + 3 \sin 2t$.

2. $x = e^{-3t} \cos t + 4e^{-3t} \sin t$.

3. $q = q_0 \cos \dfrac{t}{\sqrt{LC}}, \; i = -\dfrac{q_0}{\sqrt{LC}} \sin \dfrac{t}{\sqrt{LC}}$

5. $x = 5 \cosh 3t + \sinh 3t = 3e^{3t} + 2e^{-3t}$.

6. $x = Ae^{-at}$.

7. $x = A \cos bt + \dfrac{B}{b} \sin bt$.

8. $x = Ae^{-at} \cos bt + \dfrac{B + aA}{b} e^{-at} \sin bt$.

9. $x = A \cosh bt + \dfrac{B}{b} \sinh bt$.

10. $x = Ae^{-at} + (B + aA)te^{-at}$.

11. $\dfrac{9}{p - 3} - \dfrac{7}{p - 2}$

12. $\dfrac{\frac{1}{2}}{p - 1} - \dfrac{\frac{5}{2}}{p - 3} + \dfrac{2}{p - 4}$

13. $\dfrac{1}{2} \dfrac{a + b}{p - k} + \dfrac{1}{2} \dfrac{a - b}{p + k}$

14. $-\dfrac{3}{2} \dfrac{1}{p} + \dfrac{4}{p + 1} - \dfrac{5}{2} \dfrac{1}{p + 2}$.

15. $\dfrac{4}{p^2} + \dfrac{3}{p} - \dfrac{2}{p - 1}$.

16. $\dfrac{ac + d}{a - b} \dfrac{1}{p - a} + \dfrac{bc + d}{b - a} \dfrac{1}{p - b}$.

17. $\dfrac{ac + d}{(p - a)^2} + \dfrac{c}{p - a}$.

18. $\dfrac{\frac{1}{2}}{p - 1} + \dfrac{\frac{1}{2} - p/2}{p^2 + 1}$.

19. $\dfrac{4}{p^2} + \dfrac{-4}{p^2 + 1}$.

20. $\dfrac{5}{(p - 1)^2} + \dfrac{13}{p - 1} + \dfrac{12p + 32}{p^2 + 4}$.

21. $x = \frac{1}{2} - \frac{1}{2} \cos 2t$.

22. $x = e^{-2t} + te^{-t} - e^{-t}$.

23. $x = \frac{1}{8} \cosh 2t - \frac{1}{8} \cosh t$.

24. $x = \frac{1}{2}t^2 e^{-2t}$.

25. $x = \frac{2}{3} \sin t - \frac{1}{3} \sin 2t$.

26. $x = \frac{1}{10}e^t - \frac{1}{6}e^{-t} + \frac{1}{15}e^{-4t}$.

27. $x = \frac{1}{2} \sin t - \frac{1}{2}t \cos t$.

28. $x = 3 - 3e^{-t}$.

29. $x = 3 + 3e^{-t}$.

Exercise XXX (Pages 240 to 244)

1. $i = \frac{3}{2}(1 - \epsilon^{-2t})$.

2. $i = e_0/R(1 - \epsilon^{-Rt/L})$.

3. $i = \frac{5}{2}\epsilon^{-25,000t}$.

4. $i = e_0/R\epsilon^{-t/RC}$.

5. $i = \frac{5}{2}(\epsilon^{-100t} - \epsilon^{-300t})$.

6. $i = \epsilon^{-50t}\sin 100t$.

7. $i = \frac{1}{210}(\cos 20t - \cos 50t)$.

8. $i = -\frac{1}{400}\epsilon^{-10t} + 3/2,000\epsilon^{-30t} + 10^{-3}\cos 10t + 4 \times 10^{-3}\sin 10t$.

9. $i = 1/619,040(496\cos 2t + 40\sin 2t - 496\epsilon^{-20t}\cos 10t - 500\epsilon^{-20t}\sin 10t)$.

3. $s = \frac{2}{13} - \frac{2}{13}e^{-2t}\cos 3t - \frac{4}{39}e^{-2t}\sin 3t$.

4. $s = \frac{4}{145}(\frac{9}{2}\sin 2t - 4\cos 2t - \frac{1}{3}e^{-2t}\sin 3t + 4e^{-2t}\cos 3t)$.

6. $\theta = \frac{6}{5} - \frac{6}{5}e^{-2t}\cos t - \frac{12}{5}e^{-2t}\sin t$.

7. $\theta = e^{-2t}\sin t$.

18. $i = 5(2 - \epsilon^{-t})$.

9. $i = 2\epsilon^{-t} - 2\cos 2t + 6\sin 2t$.

20. $i = 1/2,000(1 + \epsilon^{-20t})$.

1. $i = 1/5,000(\cos 10t + 3\sin 10t - \epsilon^{-20t})$.

2. $i = 3/4,004(1 + 1,000\epsilon^{-2.002t})$.

3. $i = 10^{-4}\epsilon^{-t}\cosh\dfrac{t}{\sqrt{2}}$.

Exercise XXXI (Pages 263 to 269)

1. $f = 3xt + 3t^2 + 2\cos x$.

2. $f = 2\sin 2t + \sin x$.

3. $u = e^{x+t} + 2e^x$.

4. $f = 2x^2t - 2xt^2 + \sin 2t + 3\sin x$.

5. $f = \cos x \cos t + \cos t + \cos x - 1$.

1. $u = 0$ for $t < x$, $u = \sin(t - x)$ for $t > x$.

2. $u = 0$ for $t < x/3$, $u = 5$ for $t > x/3$.

3. $u = e^{x-t}$.

4. $u = 0$ for $t < -x$, $u = \sin(t + x)$ for $t > -x$.

7. $u = 0$ for $t > x$, $u = \sin(t - x)$ for $t < x$.

8. $u = 0$ for $t < x/v$, $u = e^{-ax/v}g\left(t - \dfrac{x}{v}\right)$ for $t > x/v$.

9. $e = 0$ for $t < x/2,000$,

   $e = \dfrac{1,000}{3}[1 - \epsilon^{-6(t-x/2,000)}]$ for $t > \dfrac{x}{2,000}$.

0. $e = 0$ for $t < x/2,000$,

   $e = \dfrac{1,000}{7}[1 - \epsilon^{-(7/3)(t-x/2,000)}]$ for $t > \dfrac{x}{2,000}$.

1. $e = 0$ for $t < x/2,000$,

   $e = 100\epsilon^{-4(t-x/2,000)}$ for $t > \dfrac{x}{2,000}$.

2. $e = 0$ for $t < x/2,000$,

   $e = 100\epsilon^{-(t-x/2,000)}\sin\left(t - \dfrac{x}{2,000}\right)$ for $t > \dfrac{x}{2,000}$.

23. $e = 0$ for $t < x/20$,

$$e = \frac{162}{17}\left[\epsilon^{-(t-x/20)} - 13\sin\left(t - \frac{x}{20}\right) + \cos\left(t - \frac{x}{20}\right)\right] \text{ for } t > \frac{x}{20}.$$

24. $e = 0$ for $t < x/v$, $e = 1$ for $t > x/v$.

25. $e = 0$ for $t < x/v$, $e = \dfrac{1}{a+1}$ for $t > x/v$.

26. $e = 0$ for $t < x/v$,

$e = 1 - \epsilon^{-a(t-x/v)}$ for $t > x/v$.

27. $e = 0$ for $t < x/v$,

$e = \epsilon^{-a(t-x/v)}$ for $t > x/v$.

32. $e = 1\left(t - \dfrac{x}{v}\right) - 1\left(t - \dfrac{2S - x}{v}\right) + 1\left(t - \dfrac{2S + x}{v}\right)$

$\qquad - 1\left(t - \dfrac{4S - x}{v}\right) + 1\left(t - \dfrac{4S + x}{v}\right) - \cdots.$

A square wave of height unity moving to the right as t changes from 0 to S/v, $2S/v$ to $3S/v$, etc., and receding as t changes from S/v to $2S/v$, $3S/v$ to $4S/v$, etc.

33. $i = \dfrac{1}{Z_K}\left[1\left(t - \dfrac{x}{v}\right) + 1\left(t - \dfrac{2S - x}{v}\right) + 1\left(t - \dfrac{2S + x}{a}\right) + \cdots.$

A square wave moving to the right, augmented by one moving to the left, etc. Thus if t exceeds NS/v, i exceeds N/Z_K.

40. $e = \dfrac{1}{2}\left[1\left(t - \dfrac{x}{v}\right) + 1\left(t - \dfrac{2S - x}{v}\right)\right]$, so that $e = 1$ for $t > \dfrac{2S}{v}$.

41. $e = \dfrac{1}{2}\left[1\left(t - \dfrac{x}{v}\right) - 1\left(t - \dfrac{2S - x}{v}\right)\right]$, so that $e = 0$ for $t > \dfrac{2S}{v}$.

42. $e = \dfrac{1}{2}\left[1\left(t - \dfrac{x}{v}\right) + \dfrac{a - 1}{a + 1}1\left(t - \dfrac{2S - x}{\cdot v}\right)\right]$,

$\qquad$ so that $e = \dfrac{a}{a + 1}$ for $t > \dfrac{2S}{v}$.

43. $i = \dfrac{1}{2Z_K}\left[1\left(t - \dfrac{x}{v}\right) - 1\left(t - \dfrac{2S - x}{v}\right)\right]$, so that $i = 0$ for $t > \dfrac{2S}{v}$.

44. $i = \dfrac{1}{2Z_K}\left[1\left(t - \dfrac{x}{v}\right) + 1\left(t - \dfrac{2S - x}{v}\right)\right]$, so that $i = \dfrac{1}{Z_K}$ for $t > \dfrac{2S}{v}$.

45. $i = \dfrac{1}{2Z_K}\left[1\left(t - \dfrac{x}{v}\right) - \dfrac{a - 1}{a + 1}1\left(t - \dfrac{2S - x}{v}\right)\right]$,

$\qquad$ so that $i = \dfrac{1}{(a + 1)Z_K}$ for $t > \dfrac{2S}{v}$.

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Numbers in a parenthesis refer to problems, the first of which begins on the page whose number immediately precedes the parenthesis. Other numbers refer to pages.

TABLE OF LAPLACE TRANSFORMS

	Function $f(t) = \text{Lap}^{-1} F(p)$	Transform $\text{Lap}\, f(t) = F(p)$
1	$c_1 f(t) + c_2 g(t)$	$c_1 F(p) + c_2 G(p)$
2	$f'(t)$	$-f(0+) + pF(p)$
3	$f''(t)$	$-f'(0+) - pf(0+) + p^2 F(p)$
4	$f^{(n)}(t)$	$-f^{(n-1)}(0+) - pf^{(n-2)}(0+) -$ $\quad \cdots - p^{n-1}f(0+) + p^n F(p)$
5	$\int_{t_0}^{t} f(u)\,du$	$\dfrac{1}{p} F(p) + \dfrac{1}{p} \int_{t_0}^{0} f(u)\,du$
6	$\int_{0}^{t} f(u)\,du$	$\dfrac{1}{p} F(p)$
7	$f'(t)$, when $f(0+) = 0$	$pF(p)$
8	$f''(t)$, when $f(0+) = 0, f'(0+) = 0$	$p^2 F(p)$
9	$f^{(n)}(t)$ when $f(0+) =$ $\quad \cdots = f^{(n-1)}(0+) = 0$	$p^n F(p)$
10	$q = \int_{t_0}^{t} i\, dt$	$\dfrac{I + q_0}{p}$
11	$e^{-at} f(t)$	$F(p + a)$
12	$g(t) = 0$ for $t < b$ $g(t) = f(t - b)$ for $t > b$	$G(p) = e^{-bp} F(p)$
13	$h(t) = \int_{0}^{t} f(u)g(t - u)\,du$ $\quad = \int_{0}^{t} f(t - u)g(u)\,du$	$H(p) = F(p)G(p)$
14	1	$\dfrac{1}{p}$

	Function $f(t) = \mathrm{Lap}^{-1} F(p)$	Transform $\mathrm{Lap}\, f(t) = F(p)$
15	e^{-at}	$\dfrac{1}{p+a}$
16	t	$\dfrac{1}{p^2}$
17	$\dfrac{t^2}{2}$	$\dfrac{1}{p^3}$
18	$\dfrac{t^n}{n!}$	$\dfrac{1}{p^{n+1}}$
19	$\sin kt$	$\dfrac{k}{p^2+k^2}$
20	$\cos kt$	$\dfrac{p}{p^2+k^2}$
21	$\sinh kt$	$\dfrac{k}{p^2-k^2}$
22	$\cosh kt$	$\dfrac{p}{}$
23	te^{-at}	
24	$\dfrac{t^n}{n!}\, e^{-at}$	
25	$e^{-at}\sin$	
26	$e^{-at}\cos$	